Number 53 / 2008
Working Paper Series
by the University of Applied Sciences of bfi Vienna

Simple Time-Varying Copula Estimation

first draft: September 2008

revised: June 2009

Wolfgang Aussenegg
Vienna University of Technology
Christian Cech
University of Applied Sciences bfi Vienna

ISSN: 1995-1469 Sponsored by the Austrian Research Promotion Agency under the *FH*plus programme

Simple Time-Varying Copula Estimation

Wolfgang Aussenegg and Christian Cech

Abstract
This article examines the ability of time-varying Gaussian and Student t copulas to accurately predict the probability of joint extreme co-movements in stock index returns. Using a sample of more than 20 years of daily return observations of the Eurostoxx 50 and Dow Jones Industrial 30 stock indices, Gaussian and Student t copulas are calibrated daily on a rolling window of the 250 most recent observations. We do not make assumptions on the functional form of the marginal distributions. Thus, the focus remains on the examination of the appropriateness of the two types of copulas. One of our findings is that there are time periods when the assumption of a Gaussian copula seems to be accurate and when the hypothesis of a Gaussian copula cannot be rejected in favor of a Student t copula. In other time periods, the hypothesis of a Gaussian copula can be rejected, as it underestimates the probability of joint extreme co-movements. This time periods of joint extreme co-movements are typically associated with a higher volatility environment and higher correlations between stock index returns. In applying a hit test to examine the ability of both copulas to predict the probability of joint strongly negative returns of both indices, we reject the null hypothesis of a Gaussian copula while the null hypothesis of a Student t copula cannot be rejected.

1 Introduction

Most models that assess daily market risk of financial portfolios like, e.g., RiskMetrics (see [29]) assume a multivariate Gaussian distribution of the risk factor changes

Wolfgang Aussenegg
Department of Finance and Corporate Control (Institute for Management Science), Vienna University of Technology, Vienna, Austria, e-mail: waussen@pop.tuwien.ac.at

Christian Cech
University of Applied Sciences bfi Vienna, Vienna, Austria, e-mail: christian.cech@fh-vie.ac.at

like, e.g., returns on stocks, commodities, zero-coupon-bonds, etc. Assuming a multivariate Gaussian distribution allows a parsimonious modeling of joint risk factor changes as their multivariate joint distribution is completely described by a vector of expected changes and the variance-covariance matrix, respectively a vector of the variances of the univariate risk factor changes and their correlation matrix.

Empirical evidence has shown, however, that very often daily asset returns are leptokurtic, displaying 'heavy tails'. This means that extreme deviations from the expected returns are more likely than implied by a normal distribution. References [26] and [8] were amongst the first to report deviations from normality for univariate return time series in the 1960s. More recent studies are, e.g., [2], [14], [16], [23], [24], [28], and [31]. Risk measures such as the one-day value-at-risk (VaR) or conditional value-at-risk (CVaR) for single instruments are typically underestimated if they are computed on the assumption of normally distributed risk factor changes while indeed they are leptokurtic. Approaches using more general distributions like the generalized hyperbolic distribution or one of its sub families such as the Normal Inverse Gaussian (see, e.g., [1]) or the Skewed t distribution (see, e.g., [13]) more accurately model single risk factors changes and can be used to overcome this problem.

Another important aspect when assessing the risk of financial portfolios is the dependence structure (the copula) between the risk factor changes. In a Gaussian world, i.e., assuming multivariate normality, this dependence structure is entirely defined by the correlation matrix. Empirical research, however, shows that the probability of extreme joint co-movements of risk factors, in particular the probability of joint excessive negative returns, is higher than implied by a Gaussian copula (i.e., by the dependence structure of a multivariate Gaussian distribution). Assuming a Gaussian copula, while the true copula assigns a higher probability mass to joint extreme movements also leads to an underestimation of risk measures. The Student t copula is a generalization of a Gaussian copula that assigns a higher probability mass to joint extreme movements than the latter and is a potential alternative to the Gaussian copula if a higher probability of joint extreme co-movements exists.

Most empirical studies on the dependence structure of market risk factor changes find that the goodness-of-fit of a Student t copula is superior to that of a Gaussian copula. Amongst these studies are, e.g., [27], [17], [6], [21], [5], [4], and [19]. Other studies, like, e.g., [30], [10], [18], [20], and [15] state that the Gaussian copula seems inappropriate to model the observed higher probability of extreme co-movements or find other copulas with a superior fit as compared to the Gaussian copula. Only few articles cannot reject the null hypothesis of the Gaussian copula being the true copula ([25]), or state that the Gaussian copula seems suitable to model financial time series returns ([9]).

The above-mentioned studies either employ the whole data-sample to calibrate various copulas and asses their goodness-of-fit or use conditional copulas to allow for time-varying copula parameters. This paper takes a different, more simple approach to allow for time-varying copulas by using a rolling window of 250 trading days for calibration. This corresponds to the (minimum) amount of trading days that financial institutions are required to use when calibrating their market risk models, according to the regulations for market risk of the Basel I and II treaties. Both a Gaussian and a Student t copula are calibrated on a daily basis.

One of the main advantages of a Student t copula as compared to other non-Gaussian copulas (like, e.g., Archimedean copulas) is that it is very flexible in its 'standard' version. It allows modeling both negative and positive dependencies and permits a 'reasonable' modeling of multivariate dependence for dimensions higher than two, as the dependence of pairs of random variables can be modeled individually. Moreover, the concept of a Student t copula will be well understood in practice, as it has a correlation matrix as one of its parameters—exactly like a Gaussian copula. Additionally, it has a further scalar parameter, the degrees of freedom ν that determine the probability of joint extreme movements. The lower the parameter ν, the higher is the probability of such extreme events.

The data base used in this article consists of daily returns of the Eurostoxx 50 and the Dow Jones Industrial 30 index from January 2^{nd}, 1987 to January 11^{th}, 2008. Using the 250 most recent return observations, both a Gaussian and a Student t copula are calibrated for every single trading day. Finally, a hit test is used to examine the accuracy of the predicted probability of joint extreme co-movements.

Our paper extends prior research in several ways. First, our copula estimation procedure reveals the existence of 'calm' and 'stormy' times. In some 'calm' periods Gaussian copulas seem to be justifiable, whereas this is not the case in more 'stormy' periods. Second, our analysis shows that when using a 'setup' that is currently used in market risk management departments of many financial institutions (i.e., a rolling window of 250 trading days as estimation basis) using a Student t copula is very promising.

Third, using (always) the last 250 trading days to calibrate copulas is a very simple and easily implemented procedure (especially in relation to conditional copulas) and generates in addition good results for Student t copulas.

Fourth, we are documenting that a Gaussian copula is not (always) appropriate as it tends to underestimate the probability of extreme co-movements. This is an important fact, as the Gaussian copula is implicitly contained in many market risk models due to the assumption of a multivariate normal distribution.

Fifth, we show that periods with a higher probability of joint extreme co-movements tend to coincide with time periods of higher volatility in the market. During such periods also the correlation between our index returns tends to be higher and Student t copulas are much more appropriate than Gaussian copulas.

And finally, our simple standard Student t copula approach has the advantage over other copulas (e.g. Archimedean copulas) and over other estimation techniques (e.g. pair-copulas or vines) that it can be interpreted more easily by practitioners. Our (simple) Student t copula approach generates two parameters: (i) a correlation matrix that can be interpreted like in a Gaussian world, and (ii) a degrees of freedom parameter controlling the probability of joint extreme co-movements.

Overall, we document that the risk management process in financial institutions can be improved by our easily implementable time-varying Student t copula approach.

The remainder of the paper is structured as follows. Section 2 gives a short introduction to copulas and the corresponding parameter estimation process. Section 3 describes our database and presents additional methodological issues in calibrating our copula models. The main empirical results are revealed in section 4 followed by concluding remarks in section 5.

2 Copula Approaches and Parameter Estimation

In the context of market risk measurement, copula approaches allow to decompose the modeling of multivariate joint distributions of risk factor changes into two separate steps: First, the modeling of the univariate risk factor changes on the one hand, and second, the modeling of their dependence structure, or copula, on the other hand. Copulas are functions that combine or couple (univariate) marginal distributions to a multivariate joint distribution. Sklar's theorem states that a n-dimensional joint distribution function $F(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ may be expressed in terms of the joint distribution's copula C and its marginal distribution functions $F_1, F_2, \ldots, F_n$ as

$$F(\mathbf{x}) = C\left(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\right), \qquad \mathbf{x} \in \mathbf{R}^n\,. \tag{1}$$

The copula function C is by itself a multivariate distribution with uniform marginal distributions on the interval $\mathbf{U}_1 = [0,1]$, $C : \mathbf{U}_1^n \to \mathbf{U}_1$.

In this paper we restrict ourselves to two elliptical copulas, the Gaussian copula and the Student t copula. The Gaussian copula is the copula that is implied by a multivariate Gaussian distribution. Any multivariate Gaussian distribution can be regarded as a set of univariate Gaussian distributions that are coupled with a Gaussian copula. Gaussian copulas are completely defined by only one parameter, the correlation matrix $\mathbf{P}$ ('capital Rho'). Student t copulas are a generalization of Gaussian copulas and correspond to the dependence structure implied by a multivariate Student t distribution. In addition to the correlation matrix $\mathbf{P}$ they have an additional scalar parameter ν, the degrees of freedom. This parameter controls the probability mass assigned to extreme joint co-movements of risk factor changes. Student t copulas assign a higher probability to joint extreme co-movements as compared to Gaussian copulas, the lower the parameter ν. A Student t copula with degrees of freedom limiting towards infinity, $\nu \to \infty$, corresponds to a Gaussian copula.[1]

The Student t copula seems a natural candidate in a first step to improve the calibration of models of joint return distributions of assets. It can be used in very much the same manner as a Gaussian copula, which nowadays is implicitly assumed by most market risk models as they often assume multivariate normality. Furthermore, the Student t copula will be well understood by practitioners that have already worked with models assuming multivariate normality, as the familiar notion of the

[1] For more information on Student t copulas see, e.g., [7].

correlation matrix is pertained and only one additional parameter is introduced.[2] This is in contrast to other widely used copulas in finance like, e.g., the BB1 copula and its two special cases, the Clayton and Gumbel copula. Their copula parameters cannot be interpreted in a similar way as those of the Student t or Gaussian copula. They are also less flexible in their 'standard' version as far as the modeling of higher dimensional dependence structures (more than two dimensions) is concerned. Here, more advanced approaches like nested copula constructions or pair-copula constructions (also referred to as vines) seem advisable (for a review of these models see, e.g., [3]).

Our copula parameters are estimated from the empirical return observations using the pseudo-log-likelihood method (see, e.g., [12]). In this approach, no assumptions on the specific functional form of the marginal distributions have to be made. Before conducting a maximum likelihood estimation, the empirical joint observations $\hat{\mathbf{x}}_t = (\hat{x}_{1,t}, \hat{x}_{2,t}, \ldots, \hat{x}_{n,t},)$, $t \in \{1, 2, \ldots, T\}$, are transformed into so-called pseudo-observations $\hat{\mathbf{u}}_t = (\hat{u}_{1,t}, \hat{u}_{2,t}, \ldots, \hat{u}_{n,t},)$

$$\hat{u}_{i,t} = \frac{1}{T+1} \sum_{s=1}^{T} \mathbf{1}_{\hat{x}_{i,s} \leq \hat{x}_{i,t}} \ , \tag{2}$$

where $\mathbf{1}_{\hat{x}_{i,s} \leq \hat{x}_{i,t}}$ is an indicator function that takes a value of 1 if $\hat{x}_{i,s} \leq \hat{x}_{i,t}$ and a value of 0 otherwise. Employing these pseudo-observations, the copula parameters are estimated via maximum likelihood estimation.

The pseudo-log likelihood approach nowadays is the most commonly used method as it achieves a better fit than methods that use correlation measures such as Kendall's tau or Spearman's rho to estimate the copula parameters (see, e.g., [11]). Another widely used method is the so-called IFM (Inference Function for Margins) method. The drawback of this approach is that the functional forms of the marginal distributions have to be assumed. [32] conduct a simulation study to assess the impact of misspecified marginal distributions and find that the errors for the copula parameter estimates can be very large if the marginal distributions are misspecified.

3 Data and Methodology

The data base consists of daily observations of the Eurostoxx 50 and the Dow Jones Industrial 30 stock indices from January 1st, 1987 to January 11th, 2008. Figure 1 shows the evolution of the two indices over this 22-year period. Figure 2 displays the corresponding daily index returns for both indices. Visually, one can see that both indices contemporaneously experienced times with high (especially 1987, 1997 to 2003) and low (especially 1993 to 1996, 2004 to 2006) volatility.

[2] Note however that the elements of a Student t copula parameter $\mathbf{P}_t$ will generally slightly differ from the elements of a Gaussian copula parameter $\mathbf{P}_G$ when the copulas are calibrated on the same data. Furthermore, a correlation of zero does *not* imply independence in the case of a Student t copula.

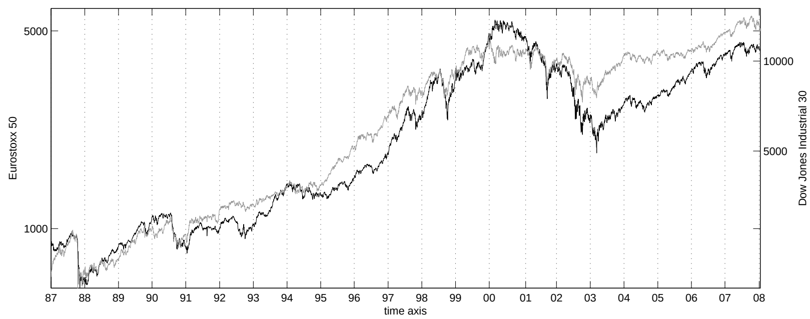

Fig. 1 This figure presents the history of the Eurostoxx 50 (left axis, black, log-scale) and the Dow Jones Industrial 30 (right axis, grey, log-scale) indices from January 1987 until January 2008. The graph shows an upward trend and indicates periods with lower and periods with higher volatility.

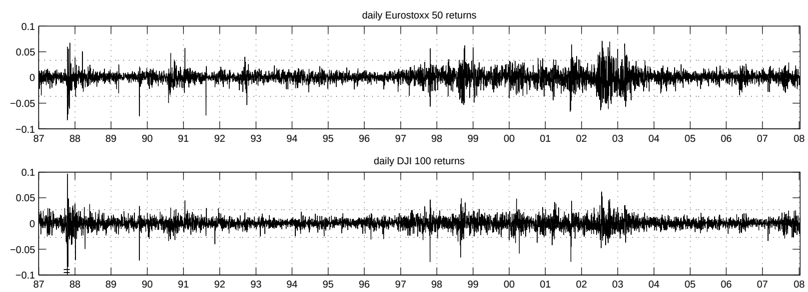

Fig. 2 This figure reveals the returns of the Eurostoxx 50 (top) and Dow Jones Industrial 30 (bottom) from January 1987 until January 2008.

Figure 3 reveals the daily returns of both indices in a scatter plot. Apparently, extreme returns tend to occur simultaneously, e.g., on ‘Black Monday’ (October 19$^{\text{th}}$, 1987), the days thereafter, during the Asian-Crisis in October 1997, the Russian Crisis in autumn 1998, or 9/11.

We use a rolling window of 250 index returns—roughly corresponding to one trading year—to estimate both the copula parameter ρ_G for a Gaussian copula and the parameters ρ_t and ν for a Student t copula for each of the 5,240 trading days from December 18$^{\text{th}}$, 1987 to January 11$^{\text{th}}$, 2008.[3]

As mentioned in the previous section, the Student t copula parameter ν is small when the data used to calibrate the copulas contains more joint extreme co-movements than implied by a Gaussian copula. And ν is large when the dependence structure between both index returns resembles a Gaussian copula. Reference [27]

[3] The parameters ρ_G and ρ_t represent the (1, 2)- and (2, 1)-elements of the two-dimensional copula parameter matrices $\mathbf{P}_G$ and $\mathbf{P}_t$, respectively. The diagonal elements of these matrices are 1.

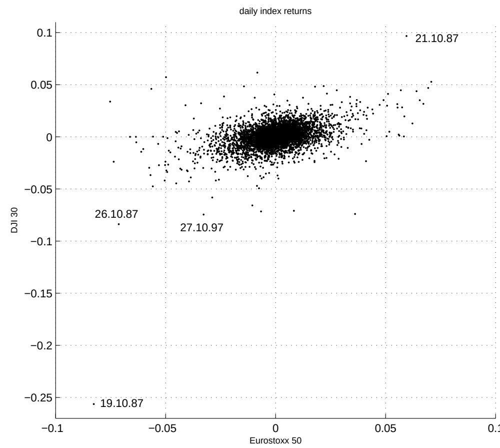

Fig. 3 This figure displays the scatter plot of the returns of the Eurostoxx 50 and Dow Jones Industrial 30 from January 1987 until January 2008

states, that Student t and Gaussian copulas are very similar for ν greater than 100. The calibration of a Student t copula (in contrast to that of a Gaussian copula) takes a much longer time as two parameters have to be estimated simultaneously in a maximum likelihood estimation that requires numerical optimization. This is particularly true when the parameter ν is large. Hence we use the following algorithm to calibrate Student t copulas:

1. Calibrate only the parameter ν and use the Gaussian copula parameter ρ_G as a proxy for ρ_t.
2. If $\nu \geq 100$, set $\nu = 100$ and calibrate the parameter ρ_t. We may interpret this Student t copula as a (quasi-) Gaussian copula.
3. If $\nu < 100$, simultaneously calibrate the parameters ν and ρ_t. Use ρ_G and ν obtained in step 1 as starting values.

This algorithm helps to speed up the calibration process considerably. Still, the calibration of a Student t copula on average takes 358 times as long as that of a Gaussian copula.[4]

4 Empirical Results

In the following two subsections we present main results of our copula parameter estimates and the hit test.

[4] The calibration on 250 empirical return observations on average takes 0.006 seconds for the Gaussian copula and 2.121 seconds for the Student t copula on a ‘standard’ personal computer.

4.1 Gaussian and Student t Copula Parameter Estimates

Figure 4 displays the time-series of the estimates of the Student t copula parameter ρ_t ('the correlation' between the two index returns)[5] and the Student t copula parameter ν (the degrees of freedom). One can observe that the correlation ρ_t varies substantially over time, taking on values between 0.119 (in 1994) and 0.677 (in 2003). Concerning the parameter ν, we can also observe substantial variability: there appear to be times, when the true copula seems to be a Gaussian copula as ν exceeds 100. In these time periods, ranging from July to October 1989, September 1994 to February 1996 and from March to July 2005, we are not able to reject the hypothesis of a Gaussian copula in favor of a Student t copula, as no high probability of joint extreme co-movements is observed in the calibration data of the preceding trading year. In other periods like, e.g., from mid December 1987 to October 1988, from August 1990 to October 1992, from September 2001 to March 2004, or from July 2006 to June 2007 the estimate of the Student t copula parameter ν is lower than 5. Hence the calibration data in these periods contain return observations that display a higher probability of joint extreme co-movements compared to a Gaussian copula. Using a likelihood ratio test, the null hypothesis of a Gaussian copula can be rejected in favor of a Student t copula in 50.5% (66.4%) of the trading days at the 1% (5%) significance level.

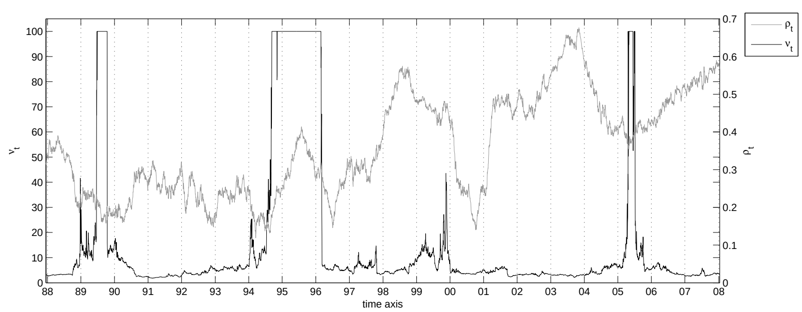

Fig. 4 This figure shows the evolution of the Student t copula parameters ν (degrees of freedom, left axis, black) and the 'correlation' parameter ρ_t (right axis, gray)

We further investigate whether there is a relationship between the magnitude of the parameter ν and the prevailing return volatility of both stock indices. Figure 5 displays the time-series of the annualized 250 day moving-average volatilities of the Eurostoxx 50 ($\sigma_t^{\text{Eurostoxx}}$) and the Dow Jones Industrial 30 indices (σ_t^{DJI}), respectively. Using linear regression, we are analyzing the following models:

[5] The Student t copula parameter ρ_t is *not* identical to the Gaussian copula parameter ρ_t , however they are of similar size. The mean absolute deviation between the two parameters amounts to 0.0126.

$$\text{Model 1}: \quad \nu_t = \beta_0 + \beta_1 \sigma_t^{\text{Eurostoxx}} + \beta_2 \sigma_t^{\text{DJI}} + \varepsilon_t \tag{3}$$

$$\text{Model 2}: \quad \nu_t = \beta_0 + \beta_1 \left(\sigma_t^{\text{Eurostoxx}} + \sigma_t^{\text{DJI}}\right) + \varepsilon_t \tag{4}$$

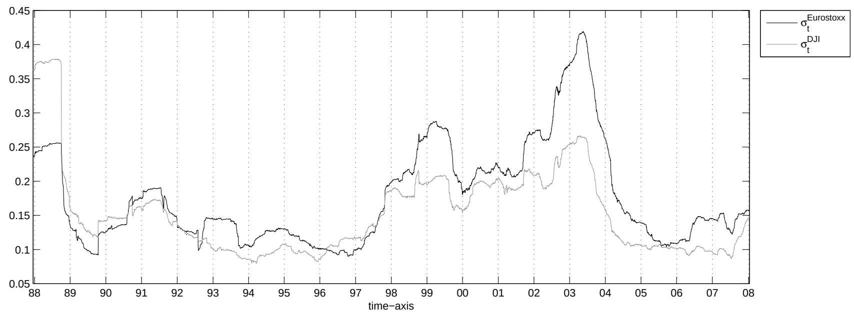

Fig. 5 This figure shows the evolution of the annualized 250-day moving average volatilities of the Eurostoxx 50 and the Dow Jones Industrial 30 index returns, assuming 250 trading days per annum

The results of the regression analysis of both models are displayed in table 1. They indicate that there is a negative relationship between the prevailing volatility of the stock indices and the copula parameter ν_t (degrees of freedom). The lower the volatility, the higher ν_t tends to be. In both models, all coefficients and the overall F-tests are statistically significant at the 1% significance level. The adjusted R^2 goodness-of-fit measure amounts to roughly 11.5%.[6]

Figure 6 reveals a scatter plot of the degrees of freedom ν_t and the regressor of model 2, $\left(\sigma_t^{\text{Eurostoxx}} + \sigma_t^{\text{DJI}}\right)$. One can identify visually that estimates of ν_t exceeding 100—indicating that the copula is a (quasi-) Gaussian copula—only occur when the return volatility of both stock indices is low.

Finally, we also investigate the relationship between the 'correlation parameter' ρ_t and the degrees of freedom ν_t, employing the following model

$$\text{Model 3}: \quad \nu_t = \beta_0 + \beta_1 \rho_{t,t} + \varepsilon_t \; . \tag{5}$$

The results for model 3 in table 1 document a negative relationship between the correlation parameter ρ_t and the degrees of freedom parameter ν_t. This negative

[6] In employing corresponding quadratic and cubic regressions, the adjusted R^2 could be augmented for model 1 to 17.5% and 21.8%, respectively, and for model 2 to 17.9% and 22.2%, respectively. The essential statement of a significant negative relationship between volatility and the Student t copula parameter ν_t (degrees of freedom) remains unchanged, however.

Table 1 The Student *t* copula parameter ν_t (degrees of freedom) is used as dependent variable. The independent variables are, in **model 1** the volatility (daily standard deviation, estimated using the past 250 daily index returns) of the Eurostoxx 50 and the Dow Jones Industrial 30 (see equation 3), in **model 2** the combined volatility (sum of the daily standard deviation of both indices, estimated using the past 250 daily index returns) of the Eurostoxx 50 and the Dow Jones Industrial 30 (see equation 4), and in **model 3** the Student *t* copula parameter ρ_t (estimated using the past 250 daily index returns) of the Eurostoxx 50 and the Dow Jones Industrial 30 (see equation 5). Coefficients significant at the 1%-level are marked bold.

Model 1	independent variables	coefficient	t-value	p-value
	Constant	**39.84**	39.365	0.000
	Standard Deviation Eurostoxx 50	**-89.29**	-11.401	0.000
	Standard Deviation Dow Jones Industrial 30	**-56.66**	-6.322	0.000
	F-value		340.82	0.000
	adjusted R-squared: 11.5%; number of observations: 5,240			

Model 2	independent variables	coefficient	t-value	p-value
	Constant	**39.87**	39.389	0.000
	SD: Eurostoxx 50 + Dow Jones Industrial 30	**-74.21**	-26.018	0.000
	F-value		676.94	0.000
	adjusted R-squared: 11.4%; number of observations: 5,240			

Model 3	independent variables	coefficient	t-value	p-value
	Constant	**30.94**	28.088	0.000
	'Correlation' parameter ρ_t	**-43.17**	-15.109	0.000
	F-value		228.27	0.000
	adjusted R-squared: 4.2%; number of observations: 5,240			

relationship is also visible in figure 7. Again, all coefficients and the overall F-test are statistically significant at the 1% significance level (see table 1).[7]

To summarize, we find that there are time periods of 'calm' markets as far as joint extreme returns are concerned, where the assumption of a Gaussian copula seems appropriate and other 'stormy' time periods with an increased probability of joint extreme co-movements. The time periods of 'calm' markets in the sense of a low probability of joint extreme co-movements tend to coincide with time periods when the stock index returns' volatility and the correlation between the index returns is low.

[7] In employing corresponding quadratic and cubic regressions, the adjusted R^2 could be augmented to 5.4% and 5.9%, respectively.

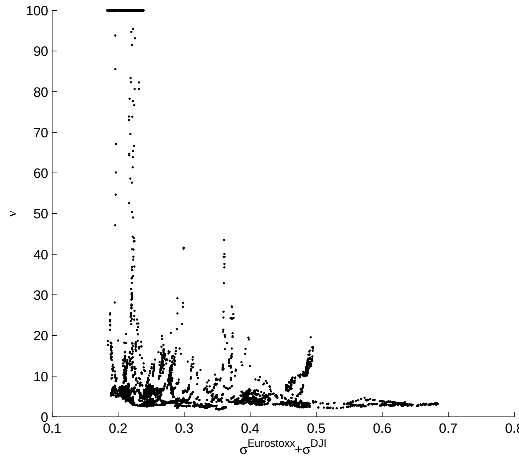

Fig. 6 Scatter plot of the Student t copula parameter ν_t (degrees of freedom) and the sum of the prevailing annualized 250-day moving-average volatilities of the Eurostoxx 50 and Dow Jones Industrial 30 index returns

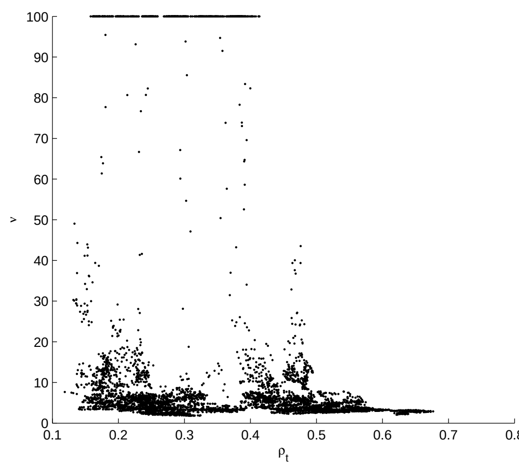

Fig. 7 Scatter plot of the Student t copula parameter ν_t (degrees of freedom) and ρ_t ('correlation')

4.2 Hit Test

Having calibrated both a Gaussian and a Student t copula for every single of the 5,240 trading days we are now interested in how good the two copulas can predict the probability of joint extreme negative returns of the two stock indices. Does it suffice to use the Gaussian copula or is it recommendable to use the more general and more complex Student t copula?

To answer this question we conduct a hit test. For every single trading day we use both the calibrated Gaussian and Student t copulas, C_G and C_t, to find the variables u_G and u_t such that

$$\mathrm{C_G}(\mathrm{u_G}, \mathrm{u_G}) = 0.01 \qquad \text{and} \qquad \mathrm{C_t}(\mathrm{u_t}, \mathrm{u_t}) = 0.01\ . \tag{6}$$

How may the values of u_G and u_t be interpreted? Consider their values obtained for the first trading day on which the copulas are calibrated, December 18$^{\text{th}}$, 1987. u_G takes a value of 0.0561 for that day while u_t takes a value of 0.0389. Recall that the probability that the Dow Jones Industrial 30 index return and the Eurostoxx 50 index return are, individually, below their 5.61%-quantile amounts to 5.61%. Assuming that the Gaussian copula is the true copula we can expect that with a 1% probability both index returns will be *jointly* below their 5.61%-quantiles. Assuming that the Student t copula is the true copula, we can expect that with a 1% probability both index returns will be below their 3.89%-quantile. From this example it can be seen how the Student t copula assigns a higher probability to joint extremely negative returns as the 3.89%-quantiles of both index return distributions are below their 5.61%-quantiles.

To inspect the obtained results of u_G and u_t, we also employ an alternative method that is purely based on the 250 most recent return pseudo-observations (equation 2) of both indices. As the data base consists of 250 observations, we know that the pseudo-observations will take values ranging from 1/251 to 250/251. In order to find the 'empirical u', u_E, we employ the following algorithm: Starting with a value of $\tilde{u} = 1/251$, we increment by steps of 1/251 until we observe (based on our 250 pairs of pseudo-observations) that

$$P\left(\hat{u}_{\text{Eurostoxx}} \leq \tilde{u} \wedge \hat{u}_{\text{DJI}} \leq \tilde{u}\right) = C_E\left(\tilde{u}, \tilde{u}\right) \geq 0.01\ , \tag{7}$$

where C_E is the empirical copula. We know that $C_E\left(\tilde{u} - 1/251, \tilde{u} - 1/251\right) < 0.01$ and may hence approximate u_E using linear interpolation:

$$u_E = \tilde{u} - \frac{1}{251} + \frac{\frac{1}{251}}{C_E\left(\tilde{u}, \tilde{u}\right) - C_E\left(\tilde{u} - \frac{1}{251}, \tilde{u} - \frac{1}{251}\right)} \times \left(0.01 - C_E\left(\tilde{u} - \frac{1}{251}, \tilde{u} - \frac{1}{251}\right)\right)\ . \tag{8}$$

Figure 8 displays the estimates of u_G, u_t and u_E over the course of time. We can observe that u_G exceeds u_t. u_t takes on values that range from 55.2% (when ν is small) to 99.13% (when ν is large) of u_G. On average, the value of u_t is 77.7% of the value of u_G. We can also observe that u_E varies substantially over time, however it is of similar magnitude as u_G and u_t. On the top of figure 8 there are three lines with cross-markers. They indicate whether the observed returns of both indices are jointly below their u_G-, u_t- and u_E-quantiles for day t, respectively. The top line indicates whether both index returns are below their u_G-quantiles, the line below whether they are both below their u_t-quantiles and the third line whether they are below their u_E-quantiles. The quantiles of the indices were computed as empirical quantiles of the 250 returns from $t - 250$ to $t - 1$, using linear interpolation.

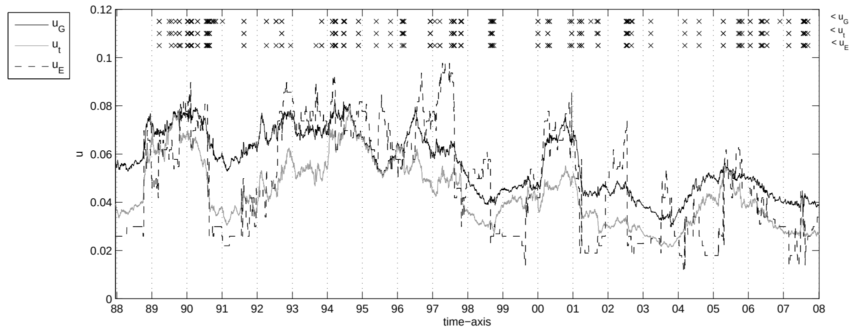

Fig. 8 This figure shows the joint return quantile for a confidence level of 1%, i.e. the joint return quantile that both indices fall on the next trading day below this return-quantile with a probability of 1%. u_G is the return quantile in using a Gaussian copula, u_t the corresponding return quantile for the Student t copula, and u_E is the corresponding empirical return quantile. Example: On December 18[th], 1987, u_G takes a value of 0.0561 and u_t takes a value of 0.0424. Assuming that the Gaussian copula is the true copula, we expect that with a 1% probability both index returns will be jointly below their 5.61%-quantiles. Assuming that the Student t copula is the true copula we expect that with a 1% probability both returns will be jointly below their 4.24%-quantiles. Every 'x' (at the top of the graph) represents a trading day on that the realized return of both indices is below the estimated threshold for that trading day (first line for u_G (Gaussian copula return quantile), second line for u_t (Student t copula return quantile), and the third line for u_E (empirical return quantile)).

If the model is correctly specified, we expect the index returns to be jointly below the thresholds in 1% of the days on average. To formally test the goodness-of-fit of the two models, i.e., the Gaussian copula model and the Student t copula model, we conduct a Kupiec hit test ([22]) that tests the null hypothesis of a correct model specification. The Kupiec test compares the number of exceptions (in the present article the number of times that both index returns are jointly below their u-quantiles) to the number of exceptions that one would expect if the models were correctly specified, i.e., $5,240 \times 0.01 = 52.4$ exceptions.

Table 2 displays in the first column the number of exceptions that can be observed for the Gaussian and Student t copula, as well as the number of exceptions for the 'empirical u'. For all three models, the number of observed exceptions is higher than the number one would expect for a correctly specified model. The second and third cloumns of table 2 contain the Kupiec test statistic LR and the corresponding p-value (i.e., the probability of committing a type 1 error if the null hypothesis of a correctly specified model is rejected). This null hypothesis cannot be rejected at any conventional statistical significance level for the model based on the Student t copula, while it can be rejected for the simple empirical model presented in equation 8 at the 5%-significance level and for the Gaussian copula model at the 1%-significance level.

Table 2 This table presents the number of exceptions, the Kupiec test statistic *LR*, and the corresponding probabilities of committing a type I error (p-value), if the null hypothesis of a correct model specification is rejected. For our total number of 5,240 trading days we would expect about 52 exceptions for a probability of 1% (i.e. a 1% probability that both indices fall below the estimated threshold for the corresponding trading day).

		Kupiec Test	
	# exceptions	*LR*	p-value
Gaussian copula	83	15.33	0.000
Student *t* copula	60	1.06	0.302
'empirical' approach	68	4.29	0.038

These findings are in line with the empirical literature on the goodness-of-fit of a Gaussian copula, that suggest that the Gaussian copula underestimates the probability of joint extreme co-movements.

5 Conclusion

This article examines the ability of Gaussian and Student *t* copulas to predict the probability of joint extreme financial returns. Using more than 5,000 daily return observations of the Eurostoxx 50 and Dow Jones Industrial 30 stock indices, time-varying Gaussian and Student *t* copulas are calibrated on a rolling window of the 250 most recent return observations.

One finding of this article is that there are times when the assumption of a Gaussian copula seems to be adequate, while there are other 'stormy' times when the hypothesis of a Gaussian copula has to be rejected in favor of a Student *t* copula. A multivariate analysis reveals that a Gaussian copula tends to be especially inappropriate in periods of higher volatility and when index returns are to a greater extent correlated.

A hit test shows that the Gaussian copula underestimates the probability of joint strongly negative returns of both indices. The hypothesis that the Gaussian copula accurately predicts the probability of joint strongly negative returns can be rejected at the 1% significance level, while for the Student *t* copula this hypothesis cannot be rejected.

Hence, the Student *t* copula seems a promising first step to improve market risk models such that the probability of joint co-movements of risk factor changes can be accurately modeled, while it will be well understood by practitioners that already know how to interpret correlation matrices.

References

1. Barndorff-Nielsen, O.E.: Normal inverse Gaussian distributions and stochastic volatility modelling. Scandinavian Journal of Statistics **24**, 1–13 (1997)
2. Bekaert, G., Harvey, C.: Emerging Equity Market Volatility. Journal of Financial Economics **43**, 29–77 (1997)
3. Berg, D., Aas, K.: Models for construction of multivariate dependence (2007). Working Paper (SAMBA/23/07), University of Oslo and Norwegian Computing Center
4. Cech, C.: An empirical investigation of the short-term relationship between interest rate risk and credit risk. In: C. Brebbia, M. Costantino, M. Larran (eds.) Computational Finance and its Applications III, pp. 185–196. WIT press (2008)
5. Chen, S., Poon, S.H.: Modelling International Stock Market Contagion Using Copula and Risk Appetite (2007). Working Paper, Manchester Business School
6. Chen, X., Fan, Y., Patton, A.: Simple Tests for Models of Dependence Between Multiple Financial Time Series, with Applications to U.S. Equity Returns and Exchange Rates (2004). London Economics Financial Markets Group Working Paper No. 483
7. Demarta, S., McNeil, A.J.: The *t* Copula and Related Copulas. International Statistical Review **73**(1), 111–129 (2005)
8. Fama, E.F.: The Behavior of Stock Market Prices. Journal of Business **3**(1), 33–105 (1965)
9. Fantazzini, D.: Dynamic Copula Modelling for Value at Risk. Frontiers in Finance and Economics **5**(2), 72–108 (2008)
10. Fortin, I., Kuzmics, C.: Tail-dependence in Stock-Return Pairs. International Journal of Intelligent Systems in Accounting, Finance and Management **11**, 89–107 (2002)
11. Genest, C., Ghoudi, K., Rivest, L.P.: A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. Biometrika **82**, 543–552 (1995)
12. Genest, C., Rivest, L.P.: Statistical Inference Procedures for Bivariate Archimedian Copulas. Journal of the American Statistical Association **88**, 1034–1043 (1993)
13. Hansen, B.E.: Autoregressive Conditional Density Estimation. Economic Review **88** (1994)
14. Hassan, M.K., Islam, A.M., Basher, S.: Market Efficiency, Time-Varying Volatility and Equity Returns in Bangladesh Stock Market (2000). Working Paper 2002-6, York University, Department of Economics
15. Hurd, M., Salmon, M., Schleicher, C.: Using Copulas to Construct Bivariate Foreign Exchange Distributions with an Application to the Sterling Exchange Rate Index (2007). Bank of England Working Paper No. 334
16. Husain, F.: The Random Walk Model in the Pakistani Equity Market: An Examination. Pakistan Development Review **36**(3), 221–240 (1997)
17. Jondeau Eric, R.M.: Conditional Dependency of Financial Series: The Copula-GARCH Model (2002). FAME Research Paper No. 69
18. Junker, M., Wagner, N., Szimayer, A.: Nonlinear Term Structure Dependence: Copula Functions, Empirics, and Risk Implications (2003). Working paper
19. Kang, L.: Modeling the Dependence Structure between Bonds and Stocks: A Multivariate Copula Approach (2008). Working paper
20. Kat, H.M., Palaro, H.P.: Replication and Evaluation of Fund of Hedge Funds Returns (2006). Alternative Investment Research Centre Working Paper No. 28
21. Kole, E., Koedijk, K., Verbeek, M.: Selecting copulas for risk management. Journal of Banking and Finance **31**(8), 2405–2423 (2007)
22. Kupiec, P.H.: Techniques for Verifying the Accuracy of Risk Management Models. Journal of Derivatives **3**, 73–84 (1995)
23. Laurence Laurence, M.M.: Weak Form Efficiency in the Kuala Lumpur and Singapore Stock Markets. Journal of Banking and Finance **10**(3), 431–445 (1986)
24. Liang, Z.H., Zhang, W., Li, S.S.: Asymmetric Extreme Dependence in Chinese Futures Markets. In: Proceedings of the International Conference on Wireless Communications, Networking and Mobile Computing, pp. 4043–4046 (2007)

25. Malevergne, Y., Sornette, D.: Testing the Gaussian Copula Hypothesis for Financial Assets Dependences (2001). Working paper
26. Mandelbrot, B.: The variation of Certain Speculative Prices. Journal of Business **36**, 394–419 (1963)
27. Mashal, R., Zeevi, A.: Beyond Correlation: Extreme Co-movements Between Financial Assets (2002). Working paper
28. Miljković, V., Radović, O.: Stylized facts of asset returns: case of BELEX. Economics and Organization **3**(2), 189–201 (2006)
29. Mina, J., Xiao, J.Y.: Return to RiskMetrics: The Evolution of a Standard (2001). RiskMetrics Group, Inc.
30. Patton, A.J.: Modeling Time-Varying Exchange Rate Dependence using the Conditional Copula (2001). UCSD Discussion Paper No. 01-09
31. Sarma, M.: Characterization of the tail behavior of financial returns: studies from India (2005). Working paper, EURANDOM
32. Scaillet, O., Fermanian, J.D.: Some statistical pitfalls in copula modeling for financial applications. In: E. Klein (ed.) Capital Formation, Governance and Banking, pp. 57–72. Nova Science Publishers (2005)

Working Papers und Studien der Fachhochschule des bfi Wien

2008 erschienene Titel

Working Papers

Working Paper Series No 42
Thomas Wala / Franz Haslehner: Was ist eine Diplomarbeit? Wien Februar 2008

Working Paper Series No 43
Vita Jagric / Timotej Jagric: Slovenian Banking Sector Experiencing the Implementation of Capital Requirements Directive. Wien Februar 2008

Working Paper Series No 44
Grigori Feiguine / Tatjana Nikitina: Die Vereinbarung Basel II – Einflüsse auf den russischen Finanzsektor. Wien Februar 2008

Working Paper Series No 45
Johannes Rosner: Die Staatsfonds und ihre steigende Bedeutung auf den internationalen Finanzmärkten. Wien März 2008

Working Paper Series No 46
Barbara Cucka: Prävention von Fraudhandlungen anhand der Gestaltung der Unternehmenskultur – Möglichkeiten und Grenzen. Wien Juni 2008

Working Paper Series No 47
Silvia Helmreich / Johannes Jäger: The Implementation and the Consequences of Basel II: Some global and comparative aspects. Vienna June 2008

Working Paper Series No 48
Franz Tödtling / Michaela Trippl: Wirtschaftliche Verflechtungen in der CENTROPE Region. Theoretische Ansätze. Wien Juni 2007

Working Paper Series No 49
Andreas Breinbauer / August Gächter: Die Nutzung der beruflichen Qualifikation von Migrantinnen und Migranten aus Centrope. Theoretische Analyse. Wien Juni 2007

Working Paper Series No 50
Birgit Buchinger / Ulrike Gschwandtner: Chancen und Perspektiven für die Wiener Wirtschaft im Kontext der Europaregion Mitte (Centrope). Ein transdisziplinärer Ansatz zur Regionalentwicklung in der Wissensgesellschaft. Eine geeschlechtsspezifische Datenanalyse. Wien Februar 2008

Working Paper Series No 51
Johannes Jäger / Bettina Köhler: Theoretical Approaches to Regional Governance. Theory of Governance. Wien Juni 2007

Working Paper Series No 52
Susanne Wurm: The Economic Versus the Social & Cultural Aspects of the European Union. Reflections on the state of the Union and the roots of the present discontent among EU citizens. Vienna September 2008

Studien

Michael Jeckle: Bankenregulierung: Säule II von Basel II unter besonderer Berücksichtigung des ICAAP. Wien Juli 2008

Alois Strobl: Pilotstudie zu: 1. Unterschiede im Verständnis des Soft Facts Rating zwischen Banken und Unternehmen
und 2. Unterschiede im Verständnis der Auswirkungen des Soft Facts Rating zwischen Banken und Unternehmen in Österreich. Wien Juli 2008

2007 erschienene Titel

Working Papers

Working Paper Series No 35
Thomas Wala / Nina Miklavc: Reduktion des Nachbesetzungsrisikos von Fach- und Führungskräften mittels Nachfolgemanagement. Wien Jänner 2007

Working Paper Series No 36
Thomas Wala: Berufsbegleitendes Fachhochschul-Studium und Internationalisierung – ein Widerspruch? Wien Februar 2007

Working Paper Series No 37
Thomas Wala / Leonhard Knoll / Stefan Szauer: Was spricht eigentlich gegen Studiengebühren? Wien April 2007

Working Paper Series No 38
Thomas Wala / Isabella Grahsl: Moderne Budgetierungskonzepte auf dem Prüfstand. Wien April 2007

Working Paper Series No 39
Thomas Wala / Stephanie Messner: Vor- und Nachteile einer Integration von internem und externem Rechungswesen auf Basis der IFRS. Wien August 2007

Working Paper Series No 40
Thomas Wala / Stephanie Messner: Synergiecontrolling im Rahmen von Mergers & Acquisitions. Wien August 2007

Working Paper Series No 41
Christian Cech: An empirical investigation of the short-term relationship between interest rate risk and credit risk. Wien Oktober 2007

Studien

Robert Schwarz: Modellierung des Kreditrisikos von Branchen mit dem Firmenwertansatz. Wien Februar 2007.

Andreas Breinbauer / Michael Eidler / Gerhard Kucera / Kurt Matyas / Martin Poiger / Gerald Reiner / Michael Titz: Kriterien einer erfolgreichen Internationalisierung am Beispiel ausgewählter Produktionsbetriebe in Ostösterreich. Wien September 2007.

2006 erschienene Titel

Working Papers

Working Paper Series No 22
Thomas Wala: Steueroptimale Rechtsform. Didactic Series. Wien Mai 2006

Working Paper Series No 23
Thomas Wala: Planung und Budgetierung. Entwicklungsstand und Perspektiven. Didactic Series. Wien Mai 2006

Working Paper Series No 24
Thomas Wala: Verrechnungspreisproblematik in dezentralisierten Unternehmen. Didactic Series. Wien Mai 2006

Working Paper Series No 25
Felix Butschek: The Role of Women in Industrialization. Wien Mai 2006

Working Paper Series No 26
Thomas Wala: Anmerkungen zum Fachhochschul-Ranking der Zeitschrift INDUSTRIEMAGAZIN. Wien Mai 2006

Working Paper Series No 27
Thomas Wala / Nina Miklavc: Betreuung von Diplomarbeiten an Fachhochschulen. Didactic Series. Wien Juni 2006

Working Paper Series No 28
Grigori Feiguine: Auswirkungen der Globalisierung auf die Entwicklungsperspektiven der russischen Volkswirtschaft. Wien Juni 2006

Working Paper Series No 29
Barbara Cucka: Maßnahmen zur Ratingverbesserung. Empfehlungen von Wirtschaftstreuhändern. Eine ländervergleichende Untersuchung der Fachhochschule des bfi Wien GmbH in Kooperation mit der Fachhochschule beider Basel Nordwestschweiz. Wien Juli 2006

Working Paper Series No 30
Evamaria Schlattau: Wissensbilanzierung an Hochschulen. Ein Instrument des Hochschulmanagements. Wien Oktober 2006

Working Paper Series No 31
Susanne Wurm: The Development of Austrian Financial Institutions in Central, Eastern and South-Eastern Europe, Comparative European Economic History Studies. Wien November 2006

Working Paper Series No 32
Christian Cech: Copula-based top-down approaches in financial risk aggregation. Wien Dezember 2006

Working Paper Series No 33
Thomas Wala / Franz Haslehner / Stefan Szauer: Unternehmensbewertung im Rahmen von M&A-Transaktionen anhand von Fallbeispielen. Wien Dezember 2006

Working Paper Series No 34
Thomas Wala: Europäischer Steuerwettbewerb in der Diskussion. Wien Dezember 2006

Studien

Andreas Breinbauer / Gabriele Bech: „Gender Mainstreaming". Chancen und Perspektiven für die Logistik- und Transportbranche in Österreich und insbesondere in Wien. Study. Wien März 2006

Johannes Jäger: Kreditvergabe, Bepreisung und neue Geschäftsfelder der österreichischen Banken vor dem Hintergrund von Basel II. Wien April 2006

Andreas Breinbauer / Michael Paul: Marktstudie Ukraine. Zusammenfassung von Forschungsergebnissen sowie Empfehlungen für einen Markteintritt. Study. Wien Juli 2006

Andreas Breinbauer / Katharina Kotratschek: Markt-, Produkt- und KundInnenanforderungen an Transportlösungen. Abschlussbericht. Ableitung eines Empfehlungskataloges für den Wiener Hafen hinsichtlich der Wahrnehmung des Binnenschiffverkehrs auf der Donau und Definition der Widerstandsfunktion, inklusive Prognosemodellierung bezugnehmend auf die verladende Wirtschaft mit dem Schwerpunkt des Einzugsgebietes des Wiener Hafens. Wien August 2006

Christian Cech / Ines Fortin: Investigating the dependence structure between market and credit portfolios' profits and losses in a top-down approach using institution-internal simulated data. Wien Dezember 2006.

2005 erschienene Titel

Working Papers

Working Paper Series No. 10
Thomas Wala: Aktuelle Entwicklungen im Fachhochschul-Sektor und die sich ergebenden Herausforderungen für berufsbegleitende Studiengänge. Wien Jänner 2005.

Working Paper Series No. 11
Martin Schürz: Monetary Policy's New Trade-Offs? Wien Jänner 2005.

Working Paper Series No. 12
Christian Mandl: 10 Jahre Österreich in der EU. Auswirkungen auf die österreichische Wirtschaft. Wien Februar 2005.

Working Paper Series No. 13
Walter Wosner: Corporate Governance im Kontext investorenorientierter Unternehmensbewertung. Mit Beleuchtung Prime Market der Wiener Börse. Wien März 2005.

Working Paper Series No. 14
Stephanie Messner: Die Ratingmodelle österreichischer Banken. Eine empirische Untersuchung im Studiengang Bank- und Finanzwirtschaft der Fachhochschule des bfi Wien. Wien April 2005.

Working Paper Series No. 15
Christian Cech / Michael Jeckle: Aggregation von Kredit und Marktrisiko. Wien Mai 2005.

Working Paper Series No. 16
Thomas Benesch / Ivancsich, Franz: Aktives versus passives Portfoliomanagement. Wien Juni 2005.

Working Paper Series No. 17
Franz Krump: Ökonomische Abschreibung als Ansatz zur Preisrechtfertigung in regulierten Märkten. Wien August 2005

Working Paper Series No. 18
Homlong, Nathalie / Springer, Elisabeth: Thermentourismus in der Ziel 1-Region Burgenland und in Westungarn als Mittel für nachhaltige Regionalentwicklung? Wien September 2005.

Working Paper Series No. 19
Wala, Thomas / Messner, Stephanie: Die Berücksichtigung von Ungewissheit und Risiko in der Investitionsrechnung. Wien November 2005.

Working Paper Series No. 20
Bösch, Daniel / Kobe, Carmen: Structuring the uses of Innovation Performance Measurement Systems. Wien November 2005.

Working Paper Series No. 21
Lechner, Julia / Wala, Thomas: Wohnraumförderung und Wohnraumversorgung in Wien. Wien Dezember 2005.

Studien

Johannes Jäger: Basel II: Perspectives of Austrian Banks and medium sized enterprises. Study. Wien März 2005.

Stephanie Messner / Dora Hunziker: Ratingmodelle österreichischer und schweizerischer Banken. Eine ländervergleichende empirische Untersuchung in Kooperation der Fachhochschule des bfi Wien mit der Fachhochschule beider Basel. Study. Wien Juni 2005.

Michael Jeckle / Patrick Haas / Christian Palmosi: Regional Banking Study. Ertragskraft-Untersuchungen 2005. Study. Wien November 2005.

2004 erschienene Titel

Working Papers

Working Paper Series No. 1
Christian Cech: Die IRB-Formel zur Berechnung der Mindesteigenmittel für Kreditrisiko. Laut Drittem Konsultationspapier und laut „Jänner-Formel" des Baseler Ausschusses. Wien März 2004.

Working Paper Series No. 2
Johannes Jäger: Finanzsystemstabilität und Basel II - Generelle Perspektiven. Wien März 2004.

Working Paper Series No. 3
Robert Schwarz: Kreditrisikomodelle mit Kalibrierung der Input-Parameter. Wien Juni 2004.

Working Paper Series No. 4
Markus Marterbauer: Wohin und zurück? Die Steuerreform 2005 und ihre Kritik. Wien Juli 2004.

Working Paper Series No. 5
Thomas Wala / Leonhard Knoll / Stephanie Messner / Stefan Szauer: Europäischer Steuerwettbewerb, Basel II und IAS/IFRS. Wien August 2004.

Working Paper Series No. 6
Thomas Wala / Leonhard Knoll / Stephanie Messner: Temporäre Stilllegungsentscheidung mittels stufenweiser Grenzkostenrechnung. Wien Oktober 2004.

Working Paper Series No. 7
Johannes Jäger / Rainer Tomassovits: Wirtschaftliche Entwicklung, Steuerwettbewerb und *politics of scale*. Wien Oktober 2004.

Working Paper Series No. 8
Thomas Wala / Leonhard Knoll: Finanzanalyse - empirische Befunde als Brennglas oder Zerrspiegel für das Bild eines Berufstandes? Wien Oktober 2004.

Working Paper Series No. 9
Josef Mugler / Clemens Fath: Added Values durch Business Angels. Wien November 2004.

<u>Studien</u>

Andreas Breinbauer / Rudolf Andexlinger (Hg.): Logistik und Transportwirtschaft in Rumänien. Marktstudie durchgeführt von StudentInnen des ersten Jahrgangs des FH-Studiengangs „Logistik und Transportmanagement“ in Kooperation mit Schenker & Co AG. Wien Frühjahr 2004.

Christian Cech / Michael Jeckle: Integrierte Risikomessung für den österreichischen Bankensektor aus Analystenperspektive. Studie in Kooperation mit Walter Schwaiger (TU Wien). Wien November 2004.

Robert Schwarz / Michael Jeckle: Gemeinsame Ausfallswahrscheinlichkeiten von österreichischen Klein- und Mittelunternehmen. Studie in Kooperation mit dem „Österreichischen Kreditschutzverband von 1870“. Wien November 2004.

Fachhochschule des bfi Wien Gesellschaft m.b.H.
A-1020 Wien, Wohlmutstraße 22
Tel.: +43/1/720 12 86
Fax.: +43/1/720 12 86-19
E-Mail: info@fh-vie.ac.at
www.fh-vie.ac.at

IMPRESSUM: Fachhochschule des bfi Wien Gesellschaft m.b.H.
Alle: A-1020 Wien, Wohlmutstraße 22, Tel.: +43/1/720 12 86