University of Applied Sciences BFI Vienna
Economics, Management and Finance

Preparation Material

Bachelor Programmes I UAS BFI Vienna

Start: 2022/23

Everything you need to know about your Admission Test

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My Future. My Education.

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I Introduction

We are delighted that you have decided to apply for a degree programme at the UAS BFI Vienna. We wish you much success for your upcoming admission test and hope to welcome you as a student at our university in the new academic year!

Please note that it is not necessary to take an external course to prepare for our admission test. This guide will ensure that you are fully prepared.

Overview of the admission procedure

Our admission procedure comprises an **online admission test** made up of a general section (coloured dark blue in the diagram below) and a degree-programme-specific section (light blue, also below).

In addition, online info sessions are offered for the individual degree programmes. We highly recommend that you attend these sessions in order to get a good idea of your chosen degree programme, to ask questions and to learn interesting facts about studying at the UAS BFI Vienna. Please use this opportunity to get to know the degree programme and its team and to clarify any questions you may have! Participation in the info sessions is voluntary and does not count towards the result of the admission test.

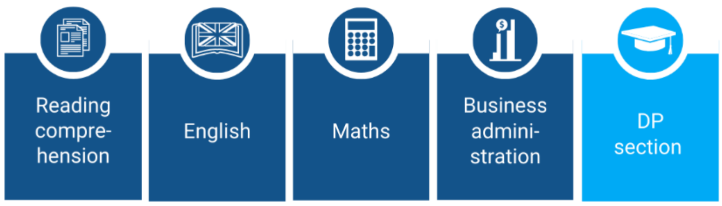
If you have applied for a German-language bachelor programme, you will find your preparation materials in the German-language reader.

This preparation guide contains study materials relating to the following test components:

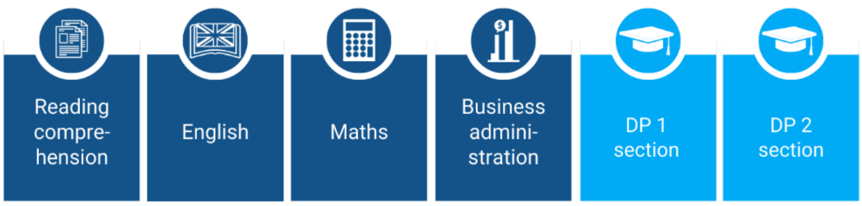
- Maths
- Business administration
- Degree-programme-specific section

Elements of the admission procedure

Applying for one degree programme (DP)



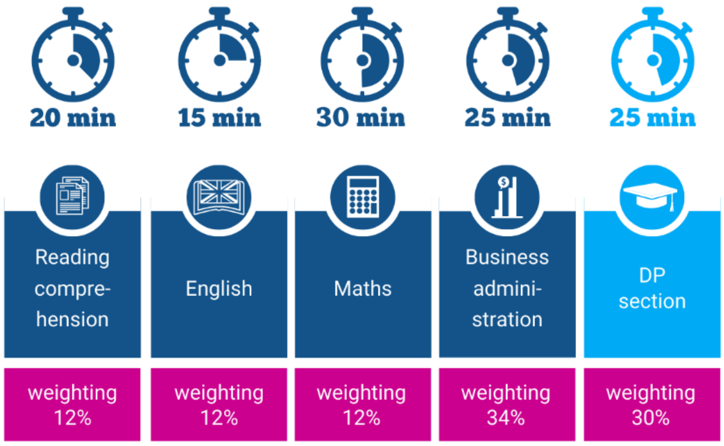
Applying for two degree programmes or more degree programmes



The admission test will be in the language of your first-choice degree programme (DP 1). If you have applied for more than one degree programme with different languages of instruction (i.e. German and English), you are required to take the reading comprehension component in both languages.

You must take a degree-programme-specific section to each of the degree programmes (light blue) you have applied for. The general test parts (text comprehension, English, mathematics, business administration - dark blue) are only taken once.

Duration and weighting of the individual test sections



If you are applying for one degree programme only, the admission test will take a total of 115 minutes, with the option of taking a break between the individual test parts. For additional degree-programme-specific parts – in case you applied for more than one programme – or text comprehension in the second language of instruction, please plan for the corresponding additional maximum working time.

You can find all information on the procedure of the admission test and the technical requirements for participation here.

Language requirements

Degree programmes taught in German

You need to have a CEFR C1 level in German and a CEFR B2 level in English to be accepted onto one of our German-language degree programmes.

Degree programmes taught in English

You need to have a CEFR C1 level in English to be accepted onto one of our English-language degree programmes. German language skills are not required.

Take the first step into your future now!

We wish you the best of luck for the admission procedure and hope to welcome you as a UAS BFI Vienna student in autumn.

II Mathematics

For the Mathematics admission test, you are allowed to use the **calculator** which is **integrated in the exam tool**, as well as a pen and some paper for taking notes. You can familiarise yourself in advance with the functionality of this calculator by taking the trial test. As soon as you have received the access data for the admission test, you will also have access to the trial test. Please make use of this possibility and note that external aids (own calculators, etc.) are **not permitted** and their use will lead to expulsion from the admission procedure!

Linear equations

Solving linear equations

While simple equations might be solved by trial and error, more complicated equations require a systematic approach. When solving for an unknown variable, the aim is to isolate the unknown on one side of the equation. This is done by applying mathematical operations to the equation. When doing so, one must ensure that any operations performed on one side of the equation are also be applied to the other side - this is called equivalent transformation.

Examples of equivalent transformation include:

- add or subtract the same number on both sides
- multiply or divide both sides by the same number (as long as this number is not 0)
- add or subtract the same multiple of the unknown on both sides

Example: $6x - 4 - 3x + 19 = 3x + 21 - 5x + 4$

We begin by simplifying the expression on each side of the equation.

$6x - 3x + 19 - 4 = 3x - 5x + 21 + 4$

$3x + 15 = -2x + 25$

We now use addition and subtraction (on both sides) to bring all occurrences of the unknown variable to one side of the equation, and all terms not involving the unknown to the other side.

$3x + 15 = -2x + 25 \qquad |+2x - 15$

$3x + 2x = 25 - 15$

$5x = 10$

We divide both sides by 5 so that we are left with only x on the left.

$5x = 10 \qquad |/5$

$x = 10/5 = 2$

Exercises: $7x - 4x + 27 = 68 - 3x + x + 4$

$6x + 2x - 36 = 4x + 54 - x$

$(x + 2) \cdot 5 + 3 \cdot (2x - 3) = 48 + (x + 12) \cdot 4 + 3$

Solutions: $x = 9$

$x = 18$

$x = 14$

Solvability of equations

Not all equations have unique solutions. An equation might have more than one solution or no solution.

$4s + 5 - s = 6 + 3s - 2$

$3s + 5 = 3s + 4 \qquad |-3s$

$5 = 4$ … this statement is false for any value of s.

The equation has <u>no solution</u>.

$4r + 5 - r = 6 + 3r - 1$

$3r + 5 = 3r + 5 \qquad |-3r$

$5 = 5$ … this statement is true for any value of r.

The equation has <u>an infinite number of solutions</u>.

Linear inequalities

An inequality tells us about the size relationship between two values, terms or expressions:

$T_L < T_R$	T_L is less than T_R
$T_L \leq T_R$	T_L is less than or equal to T_R
$T_L \geq T_R$	T_L is greater than or equal to T_R
$T_L > T_R$	T_L is greater than T_R

Equivalent transformations

The direction of an inequality $T_L < T_R$ is maintained if

1. the same value a is added or subtracted on both sides.

$T_L < T_R$ $T_L + a < T_R + a$	$\mid +a$	$T_L < T_R$ $T_L - a < T_R - a$	$\mid -a$
$3 < 5$ $7 < 9$	$\mid +4$	$3 < 5$ $-1 < 1$	$\mid -4$

2. both sides are multiplied or divided by the same value $a > 0$.

$T_L < T_R$ $a\,T_L < a\,T_R$	$\mid \cdot\, a > 0$	$T_L < T_R$ $T_L/a < T_R/a$	$\mid /a > 0$
$3 < 5$ $6 < 10$	$\mid \cdot\, 2$	$3 < 5$ $1.5 < 2.5$	$\mid /2$

Multiplication or division by a negative value changes the direction of the inequality sign.

$T_L < T_R$ $a\,T_L > a\,T_R$	$\mid \cdot\, a < 0$	$T_L < T_R$ $T_L/a > T_R/a$	$\mid /a < 0$
$3 < 5$ $-6 > -10$	$\mid \cdot\, (-2)$	$3 < 5$ $-1.5 > -2.5$	$\mid /(-2)$

The results above all apply similarly to the inequalities $T_L \leq T_R$, $T_L \geq T_R$ and $T_L > T_R$.

Example: Solve the inequality $-6x - 7(7x - 31) < 2(5 + 7x)$

$-6x - 49x + 217 < 10 + 14x$

$-55x + 217 < 10 + 14x \quad \mid -14x$

$-69x + 217 < 10 \quad \mid -217$

$-69x < -207 \quad \mid /(-69)$

$x > 3$ … we reverse the inequality sign as we have divided by a negative value

Exercises:

$2x + 2 > 3x - 4$

$6x + 7(7x + 31) \geq 2(5 - 7x)$

$3(-3x - 1) - 10x + 19 \leq 7(2 - 3x) + 12$

Solutions:

$x < 6$

$x \geq -3$

$x \leq 5$

Notation for the solution(s) to an inequality

The solution set of an inequality can be stated in two forms: set notation and interval notation.

$\boldsymbol{a \leq x \leq b}$ with $\boldsymbol{a, b \in} \mathbb{R}$: The limits a and b are included in the interval, i.e. x can take any real value from a to b (both limits are included).

Set notation: $L = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ Interval notation: $L = [a\,; b]$

$\boldsymbol{a < x < b}$ with $\boldsymbol{a, b \in} \mathbb{R}$: The limits a and b are not included in the interval, i.e. x can take any real value between a and b, excluding the values a and b.

Set notation: $L = \{x \in \mathbb{R} \mid a < x < b\}$ Interval notation: $L =]a\,; b[$

$\boldsymbol{a \leq x < b}$ with $\boldsymbol{a, b \in} \mathbb{R}$: a is included in the interval, but b is not, i.e. x can take any real value from a to (below) b.

Set notation: $L = \{x \in \mathbb{R} \mid a \leq x < b\}$ Interval notation: $L = [a\,; b[$

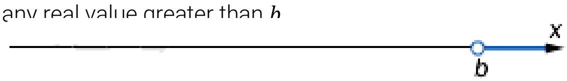
$\boldsymbol{a < x \leq b}$ with $\boldsymbol{a, b \in} \mathbb{R}$: The limit a is not included in the interval, but b is included

Set notation: $L = \{x \in \mathbb{R} \mid a < x \leq b\}$ Interval notation: $L =]a\,; b]$

$\boldsymbol{x \leq b}$ with $\boldsymbol{b \in} \mathbb{R}$: The interval has no lower limit. The upper limit b is included in the interval, and x can take any real value smaller than or equal to b.

Set notation: $L = \{x \in \mathbb{R} \mid x \leq b\}$ Interval notation: $L =]-\infty\,; b]$

$\boldsymbol{b < x}$ with $\boldsymbol{b \in} \mathbb{R}$: The limit b is not included in the interval and the interval has no upper limit, i.e. x can take any real value greater than b



Set notation: $L = \{x \in \mathbb{R} \mid b < x\}$ Interval notation: $L =]b\,; +\infty[$

The solution set of an inequality may be the entire set of real numbers $\mathbb{R}$ or the empty set { }.
- if an inequality implies a statement that is always true, then any real number is a solution: $L = \mathbb{R}$, respectively $L =]-\infty\,;\ +\infty[$
- if an inequality implies a statement that is always false, then there is no solution for the inequality: $L = \{\,\}$

Example: Provide the interval L for all solutions to the inequality from the above example:

$-6x - 7(7x - 31) < 2(5 + 7x)$

$L = \{x \in \mathbb{R} \mid x > 3\}$, respectively $L =]3\,;\ +\infty[$

Exercises: $6x + 7(7x + 31) \geq 2(5 - 7x)$

$3(-3x - 1) - 10x + 19 \leq 7(2 - 3x) + 12$

Solutions: $x \geq -3$

$x \leq 5$

Systems of linear equations

When solving for the values of more than one unknown, a single linear equation has no unique solution. There are an infinite number of value permutations that satisfy the equation. As a rule, one needs at least one independent equation per unknown variable in order to have a unique solution.

Example: Christina buys 10 units of good A and 12 units of good B.
Daniel buys 15 units of good A but only 3 units of good B.
Christina pays a total of €38 and Daniel €27.

Let the prices of goods A and B be represented by a and b respectively. These prices are unknown.
The two equations that express how the total prices for Christina and Daniel are made up are:

Christina's purchase I: $10a + 12b = 38$
Daniel's purchase II: $15a + \ \ 3b = 27$

Two methods one might use to determine the prices of goods A and B are the substitution method and the elimination method.

The substitution method

Step 1: Express one of the variables in terms of the other variable for either one of the equations.

I: $10a + 12b = 38 \quad |: 10 \quad \Rightarrow \quad a = -1.2b + 3{,}8$
II: $15a + \ \ 3b = 27$

Step 2: Substitute the expression for the chosen variable into the other equation.

II: $15a + 3b = 27 \quad \Rightarrow \quad 15(-1.2b + 3{,}8) + 3b = 27$
$-18b + 57 + 3b = 27$
$-15b = -30$
$b = 2$

Step 3: Substitute the solution obtained for the first variable into either of the equations to solve for the other variable.

I: $10a + 12b = 38 \implies 10a + 12 \cdot 2 = 38 \ \Rightarrow 10a = 14 \ |: 10 \Rightarrow a = 1.4$

Good A costs €1.40 and good B costs €2.

The elimination method

Step 1: Multiply the equations by appropriate factors so that one of the variables has the same coefficient, with opposite signs, in the two equations.

I: $10a + 12b = 38$
II: $15a + \ \ 3b = 27 \quad | \cdot (-4) \quad \Rightarrow \quad -60a - 12b = -108$

Step 2: Add the two equations

$$\begin{array}{rl} \text{I:} & 10a + 12b = 38 \\ + \quad \text{II:} & -60a - 12b = -108 \\ \hline & -50a + \ 0 \ = -70 \end{array} \quad \Rightarrow -50a = -70 \ \Rightarrow a = (-70)/(-50) = 1.4$$

Step 3: Substitute the solution obtained for the first variable into either of the equations to solve for the other unknown.

I: $10a + 12b = 38 \ \Rightarrow \ 10 \cdot 1.4 + 12b = 38 \ \Rightarrow 12b = 38 - 10 \cdot 1.4 = 24 \Rightarrow b = 24/12 = 2$

Good A costs €1.40 and good B costs €2.

Exercises:

a) I: $4x + 3y = -2$
II: $x = 3 + y$

b) I: $14x + 15y = 43$
II: $21x - 10y = 32$

c) I: $2x - y - (x - 3y) = 10$

II: $x + (y - 2) - 2x = 0$

Solutions: a) $x = 1, y = -2$
b) $x = 2, y = 1$
c) $x = 2, y = 4$

Unsolvable systems of equations

A system of equations that implies a statement that is always false has no solution, i.e. the solution set is { }.

Example: Solve the following system of equations:

I: $-4x + 6y = 8$
II: $2x - 3y = 3 \quad | \cdot 2$

$$\begin{array}{rl} \text{I:} & -4x + 6y = 8 \\ + \ \text{II:} & 4x - 6y = 6 \\ \hline & 0 = 14 \end{array}$$

… this is a false statement (i.e. it is never true), so the system of equations has no solution and the solution set is { }.

Exercises: a) I: $x + y = 5$, II: $2x + 2y = 7$
b) I: $y = -3x + 1$, II: $y = -3x + 12$

Solutions: a) There are no values of x and y that satisfy the systems of equations.
b) There are no values of x and y that satisfy the systems of equations.

If a system of equations implies a statement that is always true, then that system has an infinite number of solutions, i.e. the solution set is ℝ..

Example: Demonstrate that the system of equations below has an infinite number of solutions and provide three possible pairs of values that satisfy the equations:

I: $2x + y = 5 \quad | \cdot (-2)$
II: $4x + 2y = 10$

$$\begin{array}{rl} \text{I:} & -4x - 2y = -10 \\ + \ \text{II:} & 4x + 2y = 10 \\ \hline & 0 = 0 \end{array}$$

… this statement is always true, so the system of equations has an infinite number of solutions.

Usually, if one has the same number of linear equations as there are unknowns, then the system has at most one solution. However, if one equation in the system is a multiple of another, then there will be an infinite number of solutions.

Some of the possible solutions for this example are:

1: $x = 0 \Rightarrow 2 \cdot 0 + y = 5 \Rightarrow y = 5 - 2 \cdot 0 = 5$, i.e. $x = 0$ and $y = 5$ is a solution.
2: $x = 1 \Rightarrow 2 \cdot 1 + y = 5 \Rightarrow y = 5 - 2 \cdot 1 = 3$, i.e. $x = 1$ and $y = 3$ is a solution.
3: $x = 2 \Rightarrow 2 \cdot 2 + y = 5 \Rightarrow y = 5 - 2 \cdot 2 = 1$, i.e. $x = 2$ and $y = 1$ is a solution.

Exercises: a) I: $x - 2y = 5$, II: $-2x + 4y = -10$
b) I: $-3x + 2y = 6$, II: $9x - 6y = -18$

Solutions: a) There are an infinite number of solutions. The solution set is $\{x, y \in \mathbb{R}\}$.
b) There are an infinite number of solutions. The solution set is $\{x, y \in \mathbb{R}\}$.

Vectors

The word vector comes from the Latin for "one who carries". In physics, a vector indicates the direction and the distance of movement of an object, i.e. in which direction and how far it is "carried". Suppose the object moves three metres to the right, then five metres forwards and finally six metres upwards. Its displacement vector would then be $(3, 5, 6)$.

Definition:

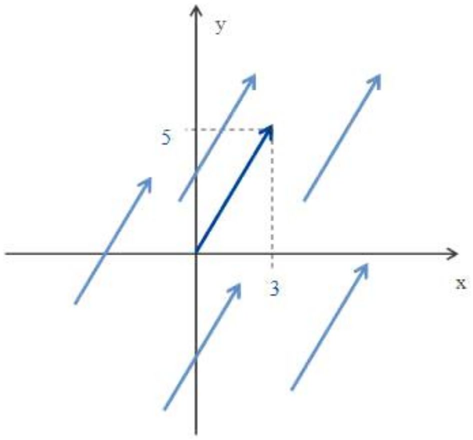
A **vector** is

- an $n \times 1$ matrix (column vector) or
- a $1 \times n$ matrix (row vector).

column vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$; row vector $\vec{x} = (x_1, \; x_2 \cdots , \; x_n)$

Example:

A two-dimensional vector indicates how many units an object moves in one direction (e.g. horizontally from left to right) and how many units it moves in a second direction (e.g. vertically from bottom to top).



All the shown vectors are $\vec{m} = \binom{3}{5}$.

The starting point is not relevant. The arrow (representing the vector) may be placed anywhere in the coordinate system.

Vectors are characterised by three properties:

- length: also called the absolute value of a vector
- direction: denotes how steep the vector is
- orientation: denotes the direction of the vector.
  If the signs of the values in the vector change (i.e. the direction of movement is reversed), then the arrowhead moves to the other end of the line.

Addition and subtraction of vectors

Definition:

Vectors are added or subtracted component by component:

$$\vec{a} \pm \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{pmatrix}$$

Scalar multiplication of a vector

Definition:

A vector $\vec{a}$ is multiplied by a number (or scalar) c by multiplying each component of the vector by c.

$$c \cdot \vec{a} = c \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} c \cdot a_1 \\ c \cdot a_2 \\ \vdots \\ c \cdot a_n \end{pmatrix}$$

Multiplication of two vectors (scalar product)

Definition:

If we multiply two (row or column) vectors $\vec{a}$ and $\vec{b}$ (both having the same number of components), then the result is not a vector but a number (or scalar). The product of two vectors is therefore called the scalar product.

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n = \sum_{i=1}^{n} a_i \cdot b_i$$

Example:

The company Turbo Oil produces refined products such as heating oil (H), diesel (D) and kerosene (K) from crude oil. The plant has an average crude oil inflow of $10t/h$ (tons per hour). The mixture produced consists of 20% heating oil, 30% diesel and 50% kerosene.

The average production quantity per hour is, therefore, H: $2t$; D: $3t$; K: $5t$.

These values can be written as a vector: $\vec{a} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

During an eight-hour shift the total quantity of used crude oil is $8 \cdot 10t = 80t$. The output quantities of the three products are as follows:

H: $8 \cdot 2t = 16t$ (20% of 80t)
D: $8 \cdot 3t = 24t$ (30% of 80t)
K: $8 \cdot 5t = 40t$ (50% of 80t)

The production quantities for the shift can also be presented in vector form. The production vector is:

$$\vec{b} = 8 \cdot \vec{a} = 8 \cdot \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \cdot 2 \\ 8 \cdot 3 \\ 8 \cdot 5 \end{pmatrix} = \begin{pmatrix} 16 \\ 24 \\ 40 \end{pmatrix}$$

With a stock of 10t for H, 15t for D and 2t for K at the beginning of the shift, the stock quantities at the end of the shift are:

H: $10t + 16t = 26t$
D: $15t + 24t = 39t$
K: $2t + 40t = 42t$

The beginning stock vector is

$$\vec{c} = \begin{pmatrix} 10 \\ 15 \\ 2 \end{pmatrix}$$

The stock quantities at the end of the shift in vector form are:

$$\vec{g} = \vec{c} + \vec{b} = \begin{pmatrix} 10 \\ 15 \\ 2 \end{pmatrix} + \begin{pmatrix} 16 \\ 24 \\ 40 \end{pmatrix} = \begin{pmatrix} 10 + 16 \\ 15 + 24 \\ 2 + 40 \end{pmatrix} = \begin{pmatrix} 26 \\ 39 \\ 42 \end{pmatrix}$$

The total sales revenue R of the stock depends on the prices for the individual products. A ton of H costs €1000, a ton of D costs €2000 and a ton of K costs €3000. The sales revenues are:

H: $€\,1000/t \cdot 26t = €26000$
D: $€\,2000/t \cdot 39t = €78000$
K: $€\,3000/t \cdot 42t = €126000$
R: $€26000 + €78000 + €126000 = €\,230000$

The price vector is

$$\vec{p} = \begin{pmatrix} 1000 \\ 2000 \\ 3000 \end{pmatrix}$$

The sales revenue R is:

$$\vec{p} \cdot \vec{g}$$
$$= \begin{pmatrix} 1000 \\ 2000 \\ 3000 \end{pmatrix} \cdot \begin{pmatrix} 26 \\ 39 \\ 42 \end{pmatrix}$$
$$= 1000 \cdot 26 + 2000 \cdot 39 + 3000 \cdot 42 = 230000$$

i.e. $R = €\,230000$

Exercise:

A company manufactures cars, lorries and motorcycles at factories at locations A, B, C and D. The production quantities and the sales prices per unit are provided in the table.

	A	B	C	D	Price per unit in €
Cars	20000	12000	4000	6000	16000
Lorries	5000	10000	2000	12000	280000
Motorcycles	8000	6000	12000	7000	4000

a) Calculate the total number of units produced for each product.

b) Calculate the company's total revenue.

Solution:

a) $\vec{g} = \begin{pmatrix} number\ of\ lorries \\ number\ of\ cars \\ number\ of\ motorcycles \end{pmatrix} = \begin{pmatrix} 42000 \\ 29000 \\ 33000 \end{pmatrix}$

b) €8924000000

Matrices

In many practical situations we perform calculations on large numbers of values. Matrix and vector notation can prove useful in such situations. Matrices can be used when solving systems of linear equations. They are also commonly used for the evaluation and optimisation of economic and technical processes.

Definition:
An $\boldsymbol{m} \times \boldsymbol{n}$ matrix is a rectangular grid of values with m rows and n columns.

a_{ij} represents the element in row i and column j of the matrix A.
e.g. a_{21} is the element in the second row and in the first column.
$m \times n$ is describes the order of the matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

Example:

Number of visitors	Monday	Wednesday	Friday	Saturday	Sunday
Adults	4	34	56	112	101
Children	60	78	24	100	123
Complimentary tickets	0	0	0	10	12

Visitor matrix

$$A = \begin{pmatrix} 4 & 34 & 56 & 112 & 101 \\ 60 & 78 & 24 & 100 & 123 \\ 0 & 0 & 0 & 10 & 12 \end{pmatrix}$$

Example:

Two companies, Gösser and Ottakringer, consume the following quantities of raw materials in the four weeks of a month:

Gösser	Hops	Malt	Water
week 1	8	4	12
week 2	10	6	5
week 3	7	8	5
week 4	11	7	9

Ottakringer	Hops	Malt	Water
week 1	6	3	12
week 2	9	5	4
week 3	7	0	5
week 4	11	6	5

Consumption matrix for Gösser

$$G = \begin{pmatrix} 8 & 4 & 12 \\ 10 & 6 & 5 \\ 7 & 8 & 5 \\ 11 & 7 & 9 \end{pmatrix}$$

Consumption matrix for Ottakringer

$$O = \begin{pmatrix} 6 & 3 & 12 \\ 9 & 5 & 4 \\ 7 & 0 & 5 \\ 11 & 6 & 5 \end{pmatrix}$$

In each of the above matrices, the rows represent the weeks, and the columns represent the raw materials. Once we understand this, the matrix notation allows for an easier comparison of the two companies' raw material quantities than a text description.
In order to compare the two companies or present aggregate information, one uses matrix subtraction and addition.

Addition and subtraction of matrices

For matrices to be added or subtracted they must have the same orders, i.e. they must have the same numbers of rows and columns.

Definition:
Two $m \times n$ matrices A and B are added or subtracted by adding or subtracting the corresponding elements.
The result is again a matrix of the order $m \times n$:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Example:

What is the aggregate consumption of raw materials for the two companies each week?

The aggregate consumption matrix is $A = G + O = \begin{pmatrix} 8 & 4 & 12 \\ 10 & 6 & 5 \\ 7 & 8 & 5 \\ 11 & 7 & 9 \end{pmatrix} + \begin{pmatrix} 6 & 3 & 12 \\ 9 & 5 & 4 \\ 7 & 0 & 5 \\ 11 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 7 & 24 \\ 19 & 11 & 9 \\ 14 & 8 & 10 \\ 22 & 13 & 14 \end{pmatrix}$

How large are the differences in consumption between the two companies per product each week?

The differences in consumption are represented by $D = G - O = \begin{pmatrix} 8 & 4 & 12 \\ 10 & 6 & 5 \\ 7 & 8 & 5 \\ 11 & 7 & 9 \end{pmatrix} - \begin{pmatrix} 6 & 3 & 12 \\ 9 & 5 & 4 \\ 7 & 0 & 5 \\ 11 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 8 & 0 \\ 0 & 1 & 4 \end{pmatrix}$

From D, we see that Gösser always uses more raw materials than Ottakringer. Four elements have the value zero, i.e. there were four occurrences of both companies using equal quantities of a raw material in the same week. Negative values in D would mean that sometimes Ottakringer uses more raw materials than Gösser.

Scalar multiplication of matrices

Definition:
A matrix is multiplied by a number (or scalar) by multiplying each element of the matrix by that number.

$$c \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} c \cdot a_{11} & c \cdot a_{12} & \cdots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \cdots & c \cdot a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \cdots & c \cdot a_{mn} \end{pmatrix}$$

Example:

What is Gösser's consumption of raw materials over five months, assuming that the same amount is consumed every month?

Gösser's consumption matrix for five months is $5 \cdot G = 5 \cdot \begin{pmatrix} 8 & 4 & 12 \\ 10 & 6 & 5 \\ 7 & 8 & 5 \\ 11 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 5 \cdot 8 & 5 \cdot 4 & 5 \cdot 12 \\ 5 \cdot 10 & 5 \cdot 6 & 5 \cdot 5 \\ 5 \cdot 7 & 5 \cdot 8 & 5 \cdot 5 \\ 5 \cdot 11 & 5 \cdot 7 & 5 \cdot 9 \end{pmatrix} = \begin{pmatrix} 40 & 20 & 60 \\ 50 & 30 & 25 \\ 35 & 40 & 25 \\ 55 & 35 & 45 \end{pmatrix}$

Matrix multiplication

Multiplication of matrices is only possible when the first matrix has as many columns as the second matrix has rows. For example, matrix A with order $m \times p$ and matrix B with order $p \times n$ can be multiplied as $A \cdot B$. The resulting matrix has the order $m \times n$.

Definition:
Two matrices of orders $(m \times p)$ and $(p \times n)$ are multiplied together by multiplying each row of the first matrix (element by element) by each column of the second matrix and adding up the resulting products.
The resulting matrix has order $(m \times n)$:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$$

Here are some examples of the calculation of c_{ij}:

$c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + \cdots + a_{1p} \cdot b_{p1}$	$c_{12} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + \cdots + a_{1p} \cdot b_{p2}$
$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + \cdots + a_{2p} \cdot b_{p1}$	$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + \cdots + a_{2p} \cdot b_{p2}$
$c_{m1} = a_{m1} \cdot b_{11} + a_{m2} \cdot b_{21} + \cdots + a_{mp} \cdot b_{p1}$	$c_{m2} = a_{m1} \cdot b_{12} + a_{m2} \cdot b_{22} + \cdots + a_{mp} \cdot b_{p2}$

Matrix multiplication is not commutative, i.e. the following generally applies: $A \cdot B \neq B \cdot A$

Example:

Gösser procures its raw materials from two suppliers (Brew Ltd. and Ale Ltd.).

The suppliers can be changed on a weekly basis. Which of the two is more attractive for Gösser?

Gösser	Brew Ltd.	Ale Ltd.
Hops	€50	€55
Malt	€136	€127
Water	€80	€79

Let us call this price matrix of raw material prices P.

$$P = \begin{pmatrix} 50 & 55 \\ 136 & 127 \\ 80 & 79 \end{pmatrix}$$

At first glance, one cannot tell which supplier is cheaper for Gösser. Brew is cheaper for hops, but more expensive for the other two raw materials.

The cost matrix is $consumption\ matrix \cdot price\ matrix$

$$= G \cdot P = \begin{pmatrix} 8 & 4 & 12 \\ 10 & 6 & 5 \\ 7 & 8 & 5 \\ 11 & 7 & 9 \end{pmatrix} \cdot \begin{pmatrix} 50 & 55 \\ 136 & 127 \\ 80 & 79 \end{pmatrix} = \begin{pmatrix} 8 \cdot 50 + 4 \cdot 136 + 12 \cdot 80 & 8 \cdot 55 + 4 \cdot 127 + 12 \cdot 79 \\ 10 \cdot 50 + 6 \cdot 136 + 5 \cdot 80 & 10 \cdot 55 + 6 \cdot 127 + 5 \cdot 79 \\ 7 \cdot 50 + 8 \cdot 136 + 5 \cdot 80 & 7 \cdot 55 + 8 \cdot 127 + 5 \cdot 79 \\ 11 \cdot 50 + 7 \cdot 136 + 9 \cdot 80 & 11 \cdot 55 + 7 \cdot 127 + 9 \cdot 79 \end{pmatrix} = \begin{pmatrix} 1904 & 1896 \\ 1716 & 1707 \\ 1838 & 1542 \\ 2222 & 2205 \end{pmatrix}$$

We now see that the costs with Ale (second column) are lower than for Brew every week (first column). Thus, Ale is the better supplier for Gösser. It should be clear that the two matrices cannot be multiplied in the order $P \cdot G$, since P has order 3×2 and G has order 4×3 and $2 \neq 4$.

Solving systems of linear equations using matrices

We already know how to solve systems of linear equations. Now let's look at solving such problems using matrices.

Example:

We solve the system of equations: I: $2x_1 + 6x_2 + x_3 = -4$

II: $3x_1 - 2x_2 + 2x_3 = 5$

III: $-x_1 + 3x_2 - x_3 = -4$

We could solve this system of equations using the substitution method:

I+III: $x_1 + 9x_2 = -8$ $\Rightarrow x_1 = -8 - 9x_2$

II+2III: $x_1 + 4x_2 = -3$ $\Rightarrow -8 - 9x_2 + 4x_2 = -3$ $\Rightarrow -5x_2 = -3 + 8 = 5 \Rightarrow \boldsymbol{x_2} = 5/(-5) = \mathbf{-1}$

I+III: $x_1 + 9 \cdot (-1) = -8$ $\Rightarrow \boldsymbol{x_1} = -8 - 9 \cdot (-1) = \mathbf{1}$

III: $-1 + 3 \cdot (-1) - x_3 = -4 \Rightarrow \boldsymbol{x_3} = -4 + 1 - 3 \cdot (-1) = \mathbf{0}$

We can also solve the above system of linear equations using matrix notation. A system of linear equations always has the form $A \cdot \vec{x} = \vec{b}$, meaning that the above system of equations can be written as follows:

$$2x_1 + 6x_2 + x_3 = -4$$

$$3x_1 - 2x_2 + 2x_3 = 5$$

$$-x_1 + 3x_2 - x_3 = -4$$

or

$\begin{pmatrix} 2 & 6 & 1 \\ 3 & -2 & 2 \\ -1 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ -4 \end{pmatrix}$. This is of the form $A \cdot \vec{x} = \vec{b}$ where $A = \begin{pmatrix} 2 & 6 & 1 \\ 3 & -2 & 2 \\ -1 & 3 & -1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -4 \\ 5 \\ -4 \end{pmatrix}$

To solve the system of equations, we transform the equation $A \cdot \vec{x} = \vec{b}$. However, matrices cannot be divided by one another.
We need A^{-1}, the inverse matrix of matrix A, so that we can rewrite the equation as $\vec{x} = A^{-1} \cdot \vec{b}$.

> Definition:
> The **identity matrix of order $\boldsymbol{n}$ x $\boldsymbol{n}$** is $\boldsymbol{I} = \begin{pmatrix} i_{11} & i_{12} & \cdots & i_{1n} \\ i_{21} & i_{22} & \cdots & i_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ i_{n1} & i_{n2} & \cdots & i_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$, i.e. $i_{rc} = 1$ if $r = c$ and $i_{rc} = 0$ if $r \neq c$.
> The **inverse matrix of a square $\boldsymbol{n}$ x $\boldsymbol{n}$ matrix $\boldsymbol{A}$ is $\boldsymbol{A^{-1}}$** for which the following applies: $\boldsymbol{A \cdot A^{-1} = A^{-1} \cdot A = I}$

The identity matrix is a square matrix with the value 1 for every element along the main diagonal and the value 0 for all other elements. It is equivalent to the number 1 (for scalars). A number x multiplied by its own reciprocal $\frac{1}{x}$ yields 1. Similarly, for matrices $A \cdot A^{-1} = A^{-1} \cdot A = I$.

The inverse matrix is determined using a calculator or a computer, as the calculation is usually very computationally intensive.

$$A \cdot A^{-1} = \begin{pmatrix} 2 & 6 & 1 \\ 3 & -2 & 2 \\ -1 & 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} -0.8 & 1.8 & 2.8 \\ 0.2 & -0.2 & -0.2 \\ 1.4 & -2.4 & -4.4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \ \checkmark$$

$$\Rightarrow \ \vec{x} = A^{-1} \cdot \vec{b} = \begin{pmatrix} -0.8 & 1.8 & 2.8 \\ 0.2 & -0.2 & -0.2 \\ 1.4 & -2.4 & -4.4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

i.e. the solution to the system of equations is $\boldsymbol{x_1 = 1}$, $\boldsymbol{x_2 = -1}$ and $\boldsymbol{x_3 = 0}$ (as before).

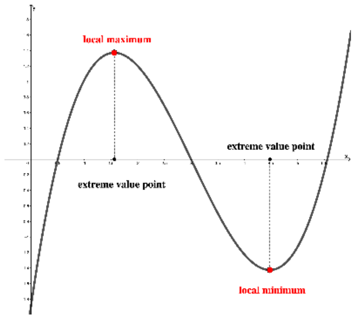
Functions

> Definition:
> A **function** is a rule which assigns to every value of the input variable a **unique** value for the output (or dependent) variable.

If the input variable is x and the output variable is y, then we could define y in terms of the function f as $y = f(x)$.
The *domain* of a function $f(x)$ is the set of all permissible input values x for $f(x)$.
The *range* of a function $f(x)$ is the complete set of all possible resulting values of $f(x)$.

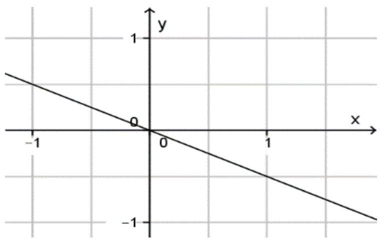
> Definition:
> The **graph** of a function $f(x)$ consists of the points $(x, f(x))$ on a two-dimensional set of axes, where x lies in the domain of $f(x)$.
> The input variable x is represented on the horizontal axis and *f(x)* (or y) is represented on the vertical axis.

> Definition:
> The **zeros** of a function $f(x)$ are those values of x for which $f(x) = 0$, i.e. the values of x at which the function graph **intersects or touches the horizontal axis.**



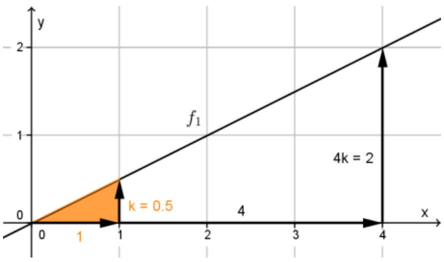
> Definition:
> An **extremum** of a function $f(x)$
> is the largest value of $f(x)$ on a given range (the **local maximum**) or on the entire domain (**global maximum**),
> or the smallest value of $f(x)$ on a given range (the **local minimum**) or on the entire domain (**global minimum**).

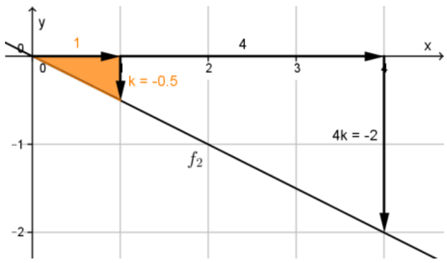
Linear functions



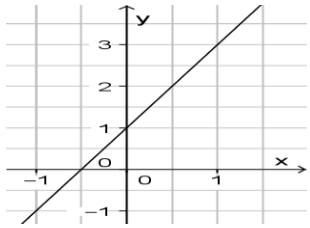
> Definition:
> A **linear function** is any function of the form $y = f(x) = kx + d$ with $k, d \in \mathbb{R}$.
> A linear function's graph is a straight line.

> Definition:
> If $y = f(x) = kx + d$, then k is the **gradient or the slope** of the linear function, i.e. $k = \frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x}$.
> If $k < 0$, then $f(x)$ is a decreasing function of x, and if $k > 0$, then $f(x)$ is an increasing function of x.





Definition:
The **vertical axis intercept** of a function $f(x)$ is the point $(0, f(0))$.
If $y = f(x) = kx + d$, then $f(0) = k \cdot 0 + d = d$,
so the vertical axis intercept of $f(x)$ is $(0, d)$.



For the graph shown, the vertical axis intercept is $(0, 1)$.

Example:

The annual number of lorry registrations in Austria increased in the years 2002 to 2006.

According to Statistik Austria, 320000 lorries were registered in 2002. In 2006 this number was 344000.

a) Describe the increase in the number of lorries using a linear model.

b) Explain the meaning of the parameters k and d in the context of this example.

c) According to the linear model, how many lorries were registered in 2005?

d) According to the linear model, in which year will the number of lorry registrations first exceed 400000?

Solution:

a) If L represents the number of registered lorries and t represents time in years (2002 represents $t = 0$), then $L(t) = kt + d$.

320000 lorries were registered in 2002 $\Rightarrow L(0) = k \cdot 0 + d = 320000 \Rightarrow d = 320000$
344000 lorries were registered in 2002; $2006 - 2002 = 4$ $\Rightarrow L(4) = k \cdot 4 + 320000 = 344000$
$\Rightarrow k = (344000 - 320000)/4 = 6000$

The linear model is $L(t) = 6000t + 320000$

b) $k = 6000$ On average, the number of registered lorries increases by 6000 every year.
$d = 320000$ At the beginning, i.e. in 2002, 320000 lorries were registered.

c) $2005 - 2002 = 3$
$L(3) = 6000 \cdot 3 + 320000 = 338000$ lorries were registered in 2005 (according to our model of linear growth).

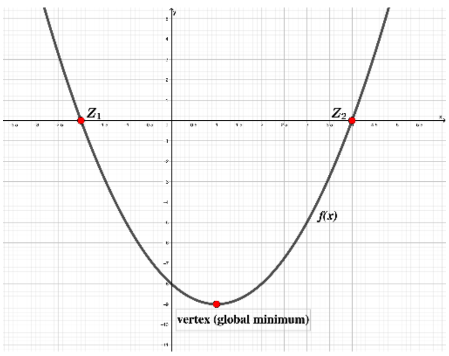
d) $L(t) = 6000t + 320000 > 400000 \quad \Rightarrow t > (400000 - 320000)/6000 = 13{,}33$
It will take 14 years for the number of lorry registrations to exceed 400000 for the first time.
This will be in the year $2002 + 14 = 2016$.

Quadratic functions

Definition:
A function of the form $y = f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a \neq 0$ is called a **quadratic function**.
The graph of a quadratic function is always a **parabola**.
The point at which the parabola reaches its maximum or minimum value is called the **vertex**.

Example:

Determine the function represented by the graph below.



We can tell the following from the diagram:

The graph is a parabola with the vertex at $(1, -9)$. This is the lowest point (minimum value) of the parabola.

The parabola is **symmetrical** about a vertical line passing through the vertex (where $x = 1$).

The parabola intersects the horizontal axis at $z_1(-2, 0)$ and $z_2(4, 0)$.
Therefore, the zeros of the function are $x_1 = -2$ and $x_2 = 4$.

The parabola intercepts the vertical axis at $(0, -8)$.

Solution:

The function is $f(x) = ax^2 + bx + c$.

$f(0) = a \cdot 0^2 + b \cdot 0 + c = \boldsymbol{c} = -\mathbf{8}$. Thus, $f(x) = ax^2 + bx - 8$

The vertex (maximum or minimum) of a quadratic function of the form $\boldsymbol{ax^2 + bx + c}$ occurs at $\boldsymbol{x = -b/2a}$.

$-b/2a = 1 \Rightarrow b = -2a$

Thus, $f(4) = a \cdot 4^2 + b \cdot 4 - 8 = 16a + 4b - 8 = 16a + 4 \cdot (-2a) - 8 = 8a - 8 = 0$ ... from z_2

$\Rightarrow 8a = 8 \Rightarrow \boldsymbol{a = 8/8 = 1}$

Thus, $\boldsymbol{b} = (-\mathbf{2}) \cdot \mathbf{1} = -\mathbf{2}$

The function is $\boldsymbol{f(x) = x^2 - 2x - 8}$

Check: $f(-2) = (-2)^2 - 2 \cdot (-2) - 8 = 0$ ... this is correct and confirms our result.

The quadratic formula

If $ax^2 + bx + c = 0$ (with a ≠ 0), then $x^2 + px + q = 0$ where $p = b/a$ and $q = c/a$.
These quadratic equations can be solved using either version of the quadratic formula:

The solutions to the equation satisfy the formula $\boldsymbol{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$, respectively $\boldsymbol{x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}}$.

The expression under the square root, $b^2 - 4ac$ and $\left(\frac{p}{2}\right)^2 - q$ respectively, is called the discriminant.

If the discriminant $\begin{cases} \textbf{... is positive, then the quadratic equation has two real solutions.} \\ \textbf{... is zero, then the quadratic equation has one real solution.} \\ \textbf{... is negative, then the quadratic equation has no real solutions.} \end{cases}$

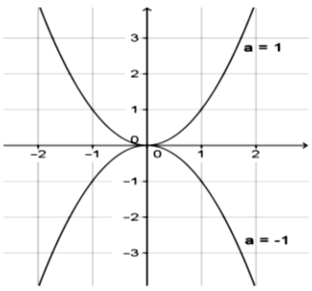
How do the values of a, b and c affect the position and shape of the quadratic function? For a and c it is easy to describe how their values affect the graph. One cannot easily draw conclusions about b, however, so it is not considered here.

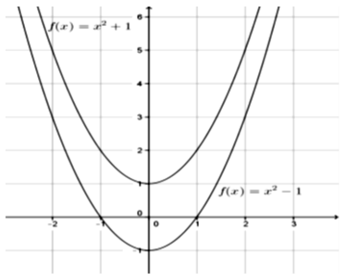
Parameter $\boldsymbol{a}$

If $a > 0$, then the graph is convex (∪) and has a minimum.

If $a < 0$, then the graph is concave (∩)and has a maximum.

As $|a|$ increases the parabola becomes narrower.





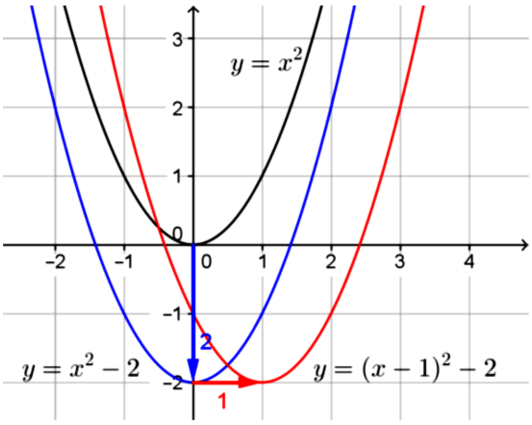
Parameter $\boldsymbol{c}$

A change in parameter c causes the graph to shift vertically (up or down) without changing its shape.

If c increases by 1, then the graph shifts upward by 1. If c reduces by 1, then the graph shifts downward by 1.

Displacement (or shifting) of functions in general.

$\boldsymbol{y = f(x) + c}$		$\boldsymbol{y = f(x + m)}$	
Vertical displacement by $\boldsymbol{c}$ units		Horizontal displacement by $\boldsymbol{m}$ units	
If $\boldsymbol{c > 0}$, then the graph shifts upward. ↑	If $\boldsymbol{c < 0}$, then the graph shifts downward. ↓	If $\boldsymbol{m > 0}$, then the graph shifts to the left. ←	If $\boldsymbol{m < 0}$, then the graph shifts to the right. →



$c = -2$: represented by the blue arrow and graph
$m = -1$: represented by the red arrow and graph

Example:

A watch manufacturer rolls out a new model and does a market analysis to determine how much profit it will make at various price points. The results are as follows: at a price of €60 the annual profit is €50000, at a price of €90 the profit is €140000 and at a price of €130 the profit is €120000.

a) Determine a quadratic model defining profit as a function of price.
b) At which price should the watch be sold to maximise the annual profit?
c) What is this maximum profit?
d) At which price does the company start to make a (positive) profit?

Solution:

a) We want a function of the form $\Pi(p) = ap^2 + bp + c$ where p represents the price of the watch in € and $\Pi(p)$ represents the annual profit in €.
We know that $\Pi(60) = 50000 \Rightarrow$ I: $a \cdot 60^2 + b \cdot 60 + c = 50000$
$\Pi(90) = 140000 \Rightarrow$ II: $a \cdot 90^2 + b \cdot 90 + c = 140000$
$\Pi(130) = 120000 \Rightarrow$ III: $a \cdot 130^2 + b \cdot 130 + c = 120000$
We can apply the elimination method to determine the solution(s) for the parameters a, b and c.
The quadratic model is: $\Pi(p) = -50p^2 + 10500p - 400000$.

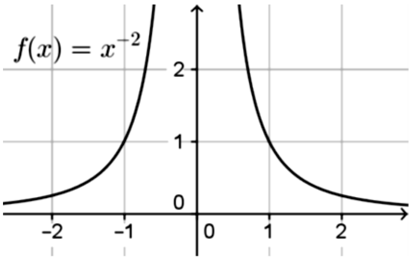
b) $a = -50 < 0$, so $\Pi(p) = -50p^2 + 10500p - 400000$ has a **maximum** value (rather than a minimum).
This maximum occurs at $p = -\frac{b}{2a} = -\frac{10500}{2\cdot(-50)} = 105$.
Thus, profit is maximised at a price of €105.

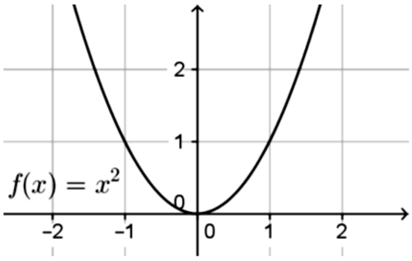
c) $\Pi(105) = -50 \cdot 105^2 + 10500 \cdot 105 - 400000 = 151250$. The maximum profit is €151250.

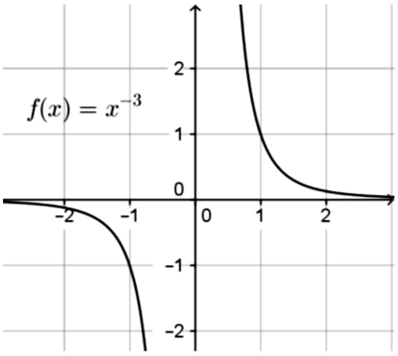
d) $\Pi(p) = -50p^2 + 10500p - 400000 = 0 \Rightarrow p_{1.2} = \frac{-10500 \pm \sqrt{10500^2 - 4\cdot(-50)\cdot(-400000)}}{2\cdot(-50)} \Rightarrow p_1 = 50$ and $p_2 = 160$.
The company first becomes profitable when the price is (higher than) €50. The company makes a (non-negative) profit for all prices from €50 to €160.

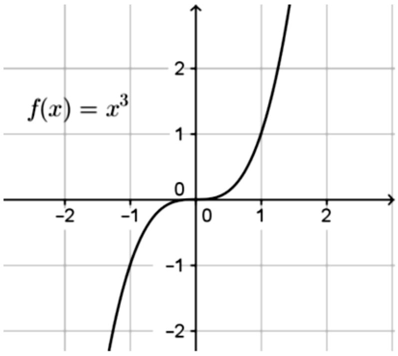
Power functions

Definition:
A **power function** is a function of the form $f(x) = cx^r$ with $c, r \in \mathbb{R}$ and $c \neq 0$.
r is called the **exponent**.









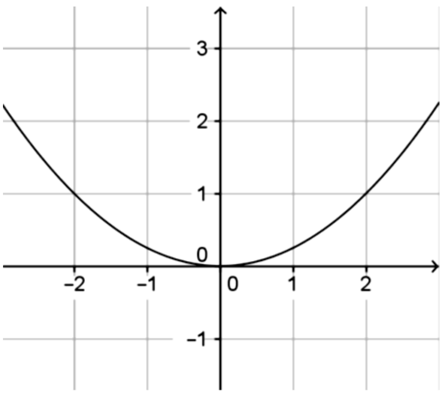
Properties of the power function:
If $f(x) = cx^r$ with $r\epsilon$ $\mathbb{Z}$ (i.e. r is an integer), then
f is **symmetrical about the vertical axis** ⇔ exponent $\boldsymbol{r}$ **is even**, and
f is **symmetrical about the origin** ⇔ exponent $\boldsymbol{r}$ **is odd**.

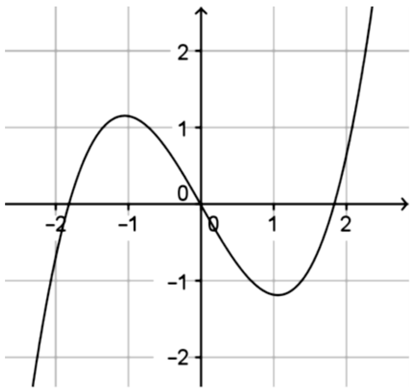
Polynomial functions

Definition:
A **polynomial function** is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ \dots\ + \ a_1 x + a_0$ with $a_i \epsilon \mathbb{R}$ and $n\epsilon\mathbb{Z}$.

Definition:
A function $f(x)$ is an **even function** if the function graph is symmetrical about the vertical axis, i.e. if $f(x) = f(-x)$ for all x.
If a polynomial function is even, then the terms with odd exponents are omitted.

A function $f(x)$ is an **odd function** if the function graph is symmetrical about the origin, i.e. if $f(x) = -f(-x)$ for all x.
If a polynomial function is odd, then the terms with even exponents are omitted.





A quadratic function, i.e. a polynomial function of the 2nd degree, is an even function. Its graph is symmetrical about the vertical axis.

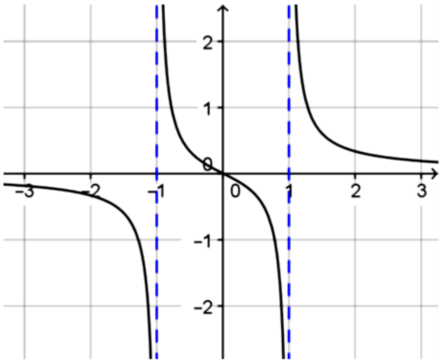
A polynomial function of the 3rd degree is an odd function. Its graph is symmetrical about the origin.

Rational functions

Definition:
A function $f(x)$ is a **rational function** if it can be represented as the quotient of two polynomial functions p and q:

$$f(x) = \frac{p(x)}{q(x)}$$

With rational functions, it is particularly important to always specify the domain.



$f(x) = x/(2(x^2 - 1))$ is an example of a rational function.

The denominator may never be equal to zero, so the domain of $f(x)$ is the set of all real numbers excluding 1 and -1. Thus, the domain is separated over three intervals on the horizontal axis.

Function $f(x)$ is not defined at the zeros of the denominator, 1 and -1.
At these values, the graph of the function approaches, but never touches, vertical lines. These lines are called **vertical asymptotes**.

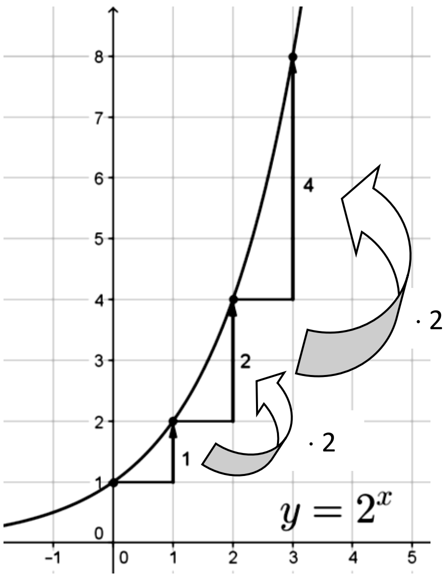
Horizontally, on the two outer intervals, the graph of the function approaches, but never touches, the horizontal axis. $f(x) = 0$ is the **horizontal asymptote**.

Functions of exponential form

Definition:
A **function of exponential form** is a function of the form $f(x) = a^x$ with $a \epsilon \mathbb{R}^+$ and $a \neq 0$.
A special case of these functions is the **exponential function** $f(x) = a^x$, where e is **Euler's number**.

Below is an example of a function of exponential form:

y = 2x



The value of the function $y = 2^x$ doubles every time x increases by 1, regardless of where x starts from. That is, the function value y doubles with increments of 1 for x.

In general, a function of the form $y = a^x$ (with $a > 1$) always increases by the same multiplicative factor for equal increments of x.

In contrast, a function of the form $y = a^{-x}$ (with $a > 1$) always **decreases** by the same multiplicative factor for equal increments of x.

A function of exponential form may also have a constant value added to it, it could have the form $f(x) = ca^x + d$ with $a \epsilon \mathbb{R}^+$,

$c, d \epsilon \mathbb{R}$ and $a, c, d \neq 0$.

Parameter $\boldsymbol{c}$

If $f(x) = ca^x + d$, then $\boldsymbol{c}$ is called the scale factor. The smaller this parameter, the more the function's graph "stretches" horizontally.
If the sign of $\boldsymbol{c}$ is changed, the curve is mirrored vertically about the horizontal line $f(x) = d$.
The graph of $f(x)$ includes the points $(0, c + d)$ and $(1, ca + d)$.

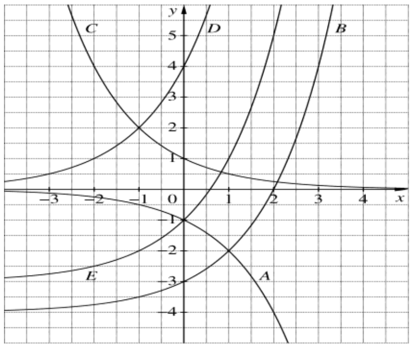
Parameter $\boldsymbol{d}$

In $f(x) = ca^x + d$ the parameter $\boldsymbol{d}$ causes vertical displacement (i.e. an upward or downward shift) of the function's graph.

Example:

Match the graphs with the appropriate function equations.

$y = 4 \cdot 2^x$
$y = 2 \cdot 2^x - 3$
$y = 2^x - 4$
$y = 0.5^x$
$y = (-1) \cdot 2^x$



Growth and decay processes

> Definition:
> Exponential growth processes can be described by the following equation: $N(t) = N_0 \cdot a^t$
> where N_0 represents the initial stock (i.e. stock at time $t = 0$)
> and a represents the growth factor ($a = 1 + p$ where p represents the periodic growth rate)
> and t represents time.
>
> If $p > 0$, then $a > 1$ and the stock grows over time (exponential growth)
> If $p < 0$, then $a < 1$ and the stock reduces over time (exponential decay)

Example:

5kg of a radioactive isotope decay exponentially over time. After five hours, 2kg of the isotope remain. What is the function describing the remaining weight of the isotope over time?

Solution:

At time $t = 0$ there are 5kg of the isotope, so $N_0 = 5$ and $N(t) = 5a^t$ where $N(t)$ is the remaining weight of the isotope at time t and t is time measured in hours.

After five hours ($t = 5$) there are 2kg remaining, so $N(5) = 5a^5 = 2 \Rightarrow a = \sqrt[5]{2/5} = 0.83255$.

The required function is $N(t) = 5 \cdot 0.83255^t$. The isotope grows at a rate of $0.83255 - 1 = -0.16745 = -16.745\%$ per hour, i.e. it decays at 16.745% per hour.

Doubling time and half-life

The doubling time and half-life are the time periods necessary for the initial stock to double and halve respectively.

Example:

The radioactive element, polonium-218, decays as described by the formula $N(t) = N_0 \cdot 0.83445^t$ (t is measured in days).

After how many days are half of the original atoms left?

Solution:

$N(t)$ describes the number of atoms at time t, so we need t for which
$N(t) = N_0 \cdot 0.83445^t = 0.5N_0 \Rightarrow 0.5 = 0.83445^t \quad \Rightarrow \ln 0.5 = t \cdot \ln 0.83445 \Rightarrow t = \ln 0{,}5 \,/\, \ln 0.83445 = 3.83$

The number of atoms will be halved after 3.83 days.

Another way of describing growth and decay processes is as follows:

> Definition:
> Exponential growth processes can be described by the following equation: $N(t) = N_0 \cdot e^{\lambda t}$
> where N_0 represents the initial stock (i.e. stock at time $t = 0$)
> and e represents Euler's number
> and λ represents the continuous rate of periodic growth
> and t represents time.
>
> If one compares the formulae $N(t) = N_0 \cdot a^t$ and $N(t) = N_0 \cdot e^{\lambda t}$, then the following becomes apparent:
> - a and e^{λ} are equal: $a = e^{\lambda} \Rightarrow \ln a = \lambda \cdot \ln e \Rightarrow \lambda = \ln a$
> - if $a > 1$, then $\lambda > 0$ (exponential growth)
>   if $0 < a < 1$, then $\lambda < 0$ (exponential decay)

Exercise:

An initial amount of €1000 at a bank earns interest at 5% per annum compounded annually.

a) What is the balance after five years?
b) How many years does it take for the balance to grow to €10000?
c) After how long has the initial amount doubled?

Solution:

a) €1276.28

b) 47.19 years. If interest is capitalised annually, then the balance will first exceed €10000 after 48 years.

c) 14.21 years. If interest is capitalised annually, then the balance will first exceed double the initial amount after 15 years.

Exercise:

In 1986, the most serious reactor disaster in the history of the civilian use of nuclear technology occurred in Chernobyl. There were high quantities of the isotopes iodine 131 and caesium 137 in the radioactive fallout, which also contaminated Austria.

a) Determine the formula for the decay of caesium 137, which has a half-life of 30 years.

b) How long does it take for caesium exposure to drop to 10% of its maximum level?

c) The law of decay for iodine 131 is $N(t) = N_0 \cdot e^{-0.08664 \cdot t}$ (t in days).
Calculate the half-life and the daily percentage decrease in iodine exposure.

d) In this case, how long does it take until only 10% of the original quantity of caesium is left?

e) A man ingests 15mg of iodine 131 with food. The iodine ends up in the thyroid gland.
After three days, he takes another 20 mg. How much iodine is still in the body a week later?

f) How long does it take after that, until there is only 1 mg left?

Solution:

a) $N^{caesium}(t) = N_0 \cdot e^{-0.0231 \cdot t}$ where t is measured in years

b) 99.66 years

c) 8 days and 8.3 %

d) 26.58 days

e) 17.21mg

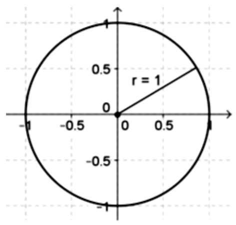
f) 32.84 days

Trigonometric functions

The sine and cosine functions

Definition:
The **unit circle** is the circle of radius 1 centred on the origin $(0, 0)$ in the Cartesian coordinate system.

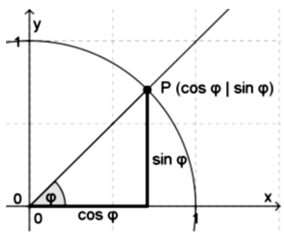
The unit circle:



Definition:
A point P lies on the unit circle. Its Cartesian coordinates depend on φ.

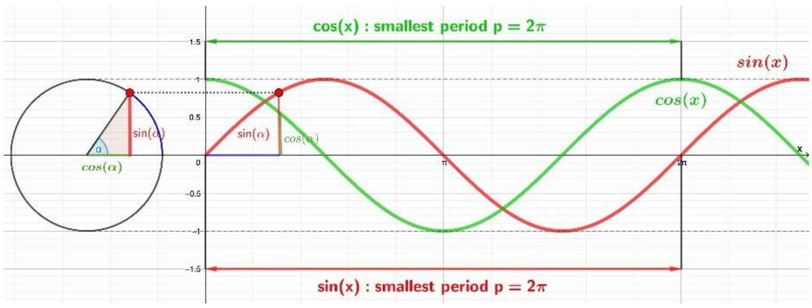
Sine:	$sin\ \varphi$	...	vertical axis coordinate of P
Cosine:	$cos\ \varphi$	...	horizontal axis coordinate of P

As the point P moves on the unit circle, the angle φ changes.



Graphing the sine and cosine functions

If we were to roll the circle from left to right, then the vertical height of the point would follow the form of the sine graph:



The cosine function has the same shape as the sine function, but it is shifted $\pi/2$ units to the right (or left).

Properties of the sine and cosine functions

1) Sine function

Zeros: The sine function has an infinite number of zeros. One of these is is at $x = \pi$.
The other zeros lie at the integer multiples of π, i.e. the zeros of the sine function are all values of $x = \pi k$ with $k \in \mathbb{Z}$.

Extrema: The sine function has an infinite number of extreme points.
Local minima: wherever $x = 3\pi/2 + 2\pi k$ with $k \in \mathbb{Z}$
Local maxima: wherever $x = \pi/2 + 2\pi k$ with $k \in \mathbb{Z}$

Symmetry: The sine function is an odd function, i.e. $\sin(x) = -\sin(-x)$ for all x.

Periodicity: The sine function is periodic with the smallest period $p = 2\pi$. The period is the distance between two consecutive points on the curve having the same (vertical) value.

2) <u>Cosine function</u>

Zeros: The cosine function has an infinite number of zeros. One of these is at $x = \frac{\pi}{2}$.
The other zeros lie at the integer multiples of $\pi/2$, i.e. wherever $x = \frac{\pi k}{2}$ with $k \in \mathbb{Z}$.

Extrema: The cosine function has an infinite number of extreme points.
Local minima: wherever $x = \pi + 2\pi k$ with $k \in \mathbb{Z}$
Local maxima: wherever $x = 2\pi + 2\pi k$ with $k \in \mathbb{Z}$

Symmetry: The cosine function is an even function, i.e. $\cos(x) = \cos(-x)$ for all x.

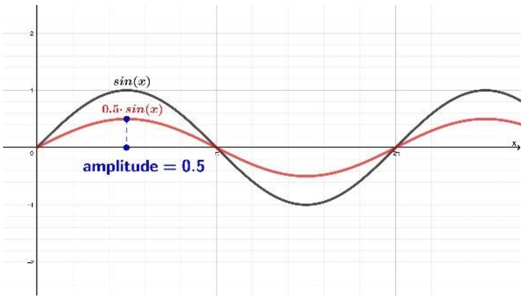
Periodicity: The cosine function is periodic with the smallest period $p = 2\pi$.

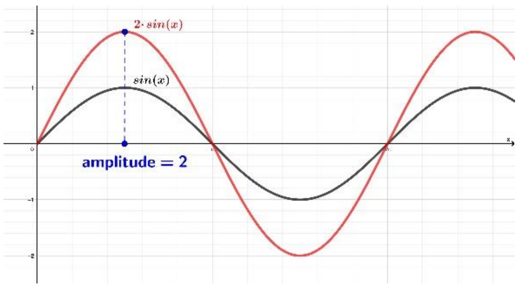
> Definition:
> Any sine function can be expressed in terms of a cosine function and vice versa, since it is the same function displaced horizontally by $\frac{\pi}{2}$ units: $\mathbf{sin}(\boldsymbol{x}) = \mathbf{cos}\left(\boldsymbol{x} - \frac{\pi}{2}\right)$ and $\mathbf{cos}(\boldsymbol{x}) = \mathbf{sin}(\boldsymbol{x} + \frac{\pi}{2})$ for all $\boldsymbol{x} \in \mathbb{R}$

Parameters for the sine and cosine functions

Definition: A general sinusoidal function has the form $f(x) = a\sin(bx + c) + d$.	
a is the amplitude b	Changes in a cause compression or stretching of the graph along the vertical axis. Changes in b cause compression or stretching of the graph along the horizontal axis.
c d	Changes in c cause displacement along the horizontal axis. Changes in d cause displacement along the vertical axis.

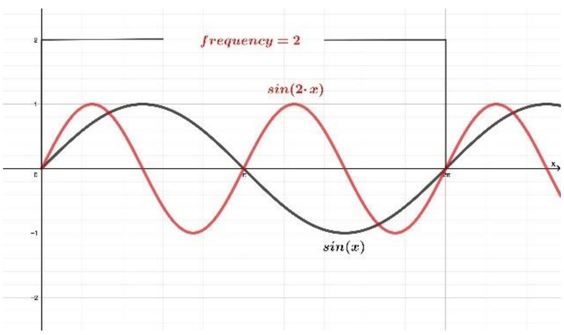
<u>The effect of changes in parameter a</u>

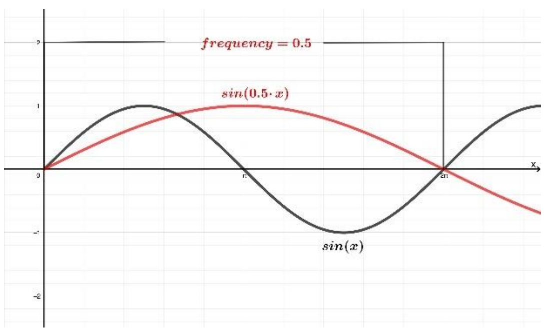




For $a > 1$, the graph is stretched vertically (the extrema lie further from the horizontal axis).
For $0 < a < 1$, the graph is compressed vertically (the extrema lie closer to the horizontal axis).
For $a < 0$, the graph is mirrored about the horizontal line $f(x) = d$.

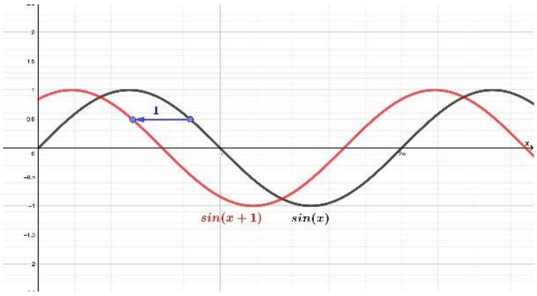
<u>Changes to parameter b</u>

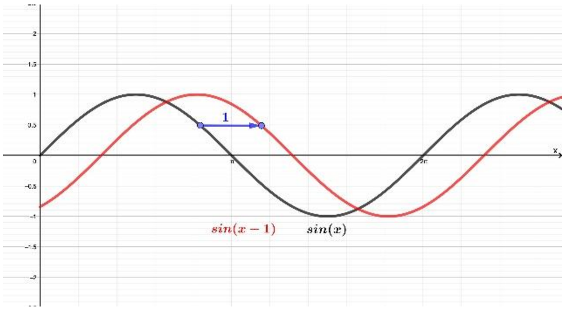




For $b > 1$, the graph is compressed horizontally, i.e. the horizontal distance between peaks reduces.
For $0 < b < 1$, the graph is stretched horizontally, i.e. the horizontal distance between peaks increases.

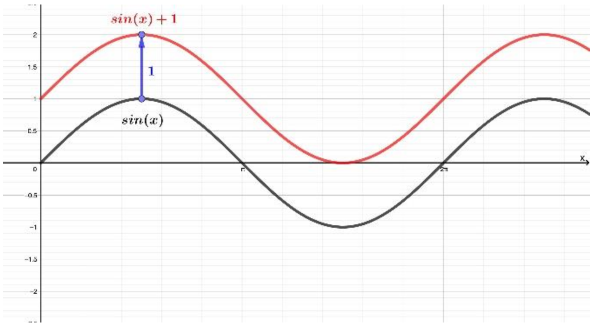
<u>Changes to parameter c</u>

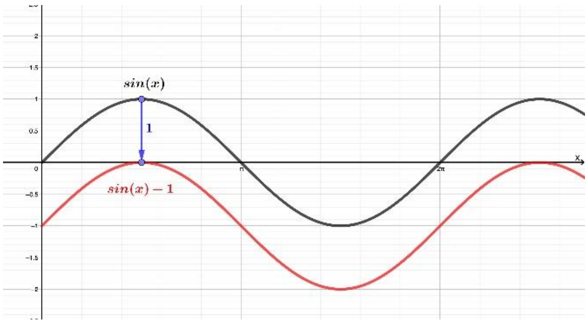




For $c > 0$, the graph is displaced to the left along the horizontal axis.
For $c < 0$, the graph is displaced to the right along the x-axis.

<u>Changes to parameter d</u>

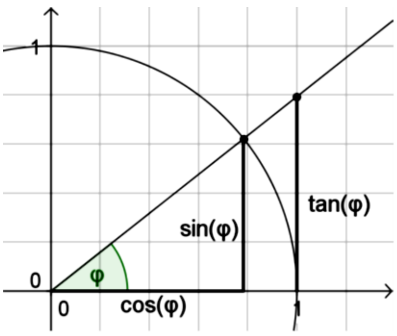




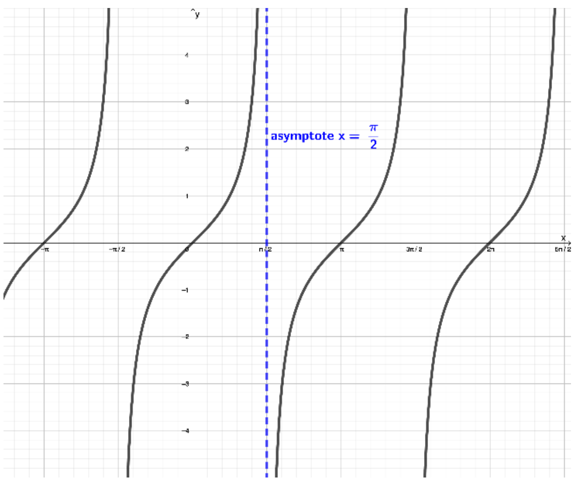
For $d > 0$, the graph is displaced upward.
For $d < 0$, the graph is displaced downward.

The tangent function

Definition:
The **tangent function** is $\tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)}$



We again consider the unit circle to better understand the tangent function. The tangent function is not defined at 90°: the sine of 90° is 1, the cosine of 90° is 0, so the tangent at 90° is $1/0$ which is not defined.



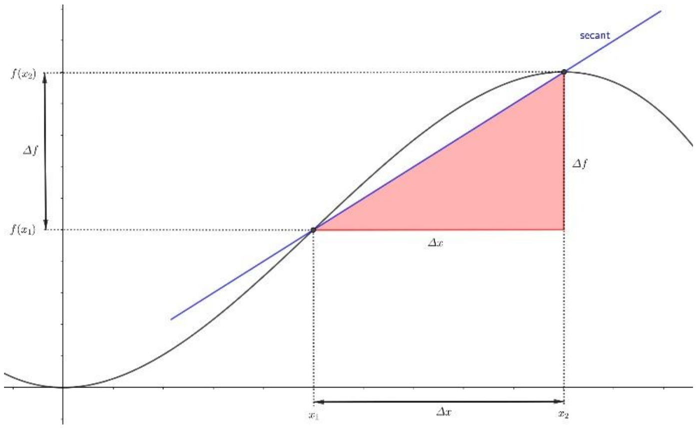
Properties of the tangent function

Zeros: wherever $x = \pi k \ (k \in \mathbb{Z})$
Asymptotes: wherever $x = \frac{\pi}{2} + \pi k \ (k \in \mathbb{Z})$
Symmetry: odd function, i.e. $\tan(x) = -\tan(-x)$ for all x
Periodicity: smallest period $p = \pi$

Differential calculus

The difference quotient

The difference quotient measures the average rate of change of the dependent variable over an interval for the independent variable. It corresponds to the gradient of the secant.

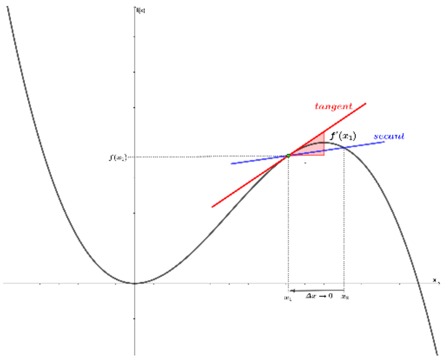


The average gradient k between x_1 and x_2 corresponds to the gradient of the secant.

The **difference quotient** for a function $f(x)$ on the interval $(x_1;\ x_2)$ is $k = \frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

The derivative

The derivative of a function corresponds to the gradient of the tangent at a point. The derivative measures the instantaneous rate of change of the function.



As Δx tends toward zero, the secant moves closer and closer to the tangent to the function at $(x_1, f(x_1))$.

The gradient of $f(x)$ at the point $(x_1, f(x_1))$ is the same as the slope of the tangent at that point.

This slope at $(x_1, f(x_1))$ is measured by the derivative, denoted as $f'(x_1)$.

The derivative of $f(x)$ is defined as $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$.

What is the slope of $f(x) = x^2$ when $x = 2$?

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2-x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}$$

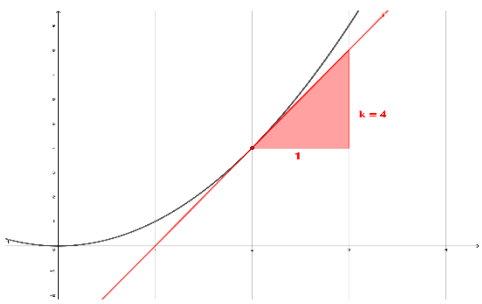
$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x$$

The derivative of $f(x) = x^2$ when $x = 2$, is $f'(2) = 2 \cdot 2 = 4$.

i.e. the function has a gradient of 4 when $x = 2$.



Rules of differentiation

In practice, we don't always use the definition formula of the derivative to determine the derivative. There are rules for taking the derivatives of various functions that make life easier.

Constant rule

If $f(x) = c$, i.e. the function has a constant value for all x, then $f'(x) = 0$ for all x.

Example: $g(x) = 2 \Rightarrow g'(x) = 0$

Power rule

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

Example: $h(x) = x^3 \Rightarrow h'(x) = 3x^2$

Factor rule

If $f(x) = cg(x)$, then $f'(x) = cg'(x)$

Example: $m(x) = 4x^3 \Rightarrow m'(x) = 4 \cdot 3x^2 = 12x^2$

Sum rule

If $f(x) = h(x) + g(x)$, then $f'(x) = h'(x) + g'(x)$

Example: $f(x) = 2x^2 + 4x + 1 \Rightarrow f'(x) = 2 \cdot 2x^1 + 4 + 0 = 4x + 4$

Product rule

If $f(x) = h(x) \cdot g(x)$, then $f'(x) = h'(x) \cdot g(x) + h(x) \cdot g'(x)$

Example: $n(x) = (3x + 4)(x^2 - 3)$ … $h(x) = 3x + 4$ and $g(x) = x^2 - 3$, so $h'(x) = 3 + 0 = 3$ and $g'(x) = 2x - 0 = 2x$
$\Rightarrow n'(x) = 3(x^2 - 3) + (3x + 4) \cdot (2x - 0) = 3x^2 - 9 + 2x(3x + 4) = 3x^2 - 9 + 6x^2 + 8x = 9x^2 + 8x - 9$

Quotient rule

If $f(x) = h(x)/g(x)$, then $f'(x) = \frac{h'(x) \cdot g(x) - h(x) \cdot g'(x)}{g(x)^2}$

Example: $p(x) = \frac{2x+1}{3x-5}$ … $h(x) = 2x + 1$ and $g(x) = 3x - 5$ so $h'(x) = 2$ and $g'(x) = 3$
$\Rightarrow p'(x) = \frac{2(3x-5) - (2x+1) \cdot 3}{(3x-5)^2} = \frac{6x - 10 - 6x - 3}{(3x-5)^2} = \frac{-13}{(3x-5)^2}$

Chain rule

If $f(x) = h(g(x))$, then $f'(x) = h'\big(g(x)\big) \cdot g'(x)$

Example: $q(x) = (x^2 + 7x)^9$ … $h(x) = x^9$ and $g(x) = x^2 + 7x$, so $h'(x) = 9x^8$ and $g'(x) = 2x + 7$
$\Rightarrow q'(x) = 9(x^2 + 7x)^8(2x + 7)$

Derivatives of other functions

The exponential function with the base e remains unchanged during differentiation, i.e. $f(x) = e^x \Rightarrow f'(x) = e^x$.

$$f(x) = \sin(x) \qquad \Rightarrow \qquad f'(x) = \cos(x)$$

The derivatives of other functions are as follows:

$$f(x) = \cos(x) \quad \Rightarrow \quad f'(x) = -\sin(x)$$

$$f(x) = \tan(x) \quad \Rightarrow \quad f'(x) = \frac{1}{\cos^2(x)}$$

$$f(x) = \ln(x) \quad \Rightarrow \quad f'(x) = \frac{1}{x}$$

Exercises

a) $f(x) = 3x^7 + 11x^5 - 8x^3 - 7x + 9$

b) $f(x) = (3x^2 - 5)(x^2 + 3x)$

c) $f(x) = 3x^2 \cdot e^x$

d) $f(x) = \frac{(3x^3 - 4x^2)}{x^2}$

e) $f(x) = (x^2 - 9)^3$

f) $f(x) = \cos(2\text{x}) \cdot x^2$

Solutions

a) $f'(x) = 21x^6 + 55x^4 - 24x^2 - 7$

b) $f'(x) = 12x^3 + 27x^2 - 10x - 15$

c) $f'(x) = 6x \cdot e^x + 3x^2 \cdot e^x$

d) $f'(x) = 3$

e) $f'(x) = 3 \cdot (x^2 - 9)^2 \cdot 2x = 6x^5 - 108x^3 + 484x$

f) $f'(x) = -2\sin(2x) \cdot x^2 + \cos(2x) \cdot 2x$

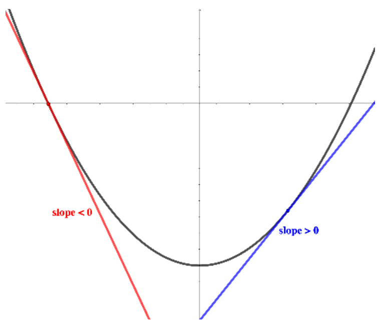
Monotonicity and the local extrema

One can use derivatives to determine extrema of functions.

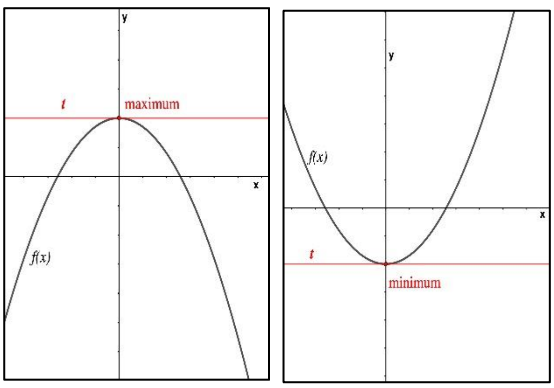
The derivative of a function can be interpreted as the gradient of the tangent to the function.

Monotonicity

A function grows strictly monotonically over an interval if
$f'(x) > 0$ for all x in the interval (**strictly increasing**)
or
$f'(x) < 0$ for all x in the interval (**strictly decreasing**).



The monotonicity of a function changes, i.e. the sign of the derivative changes, at a maximum or minimum of the function. Thus, at the extreme itself, the derivative (slope) of the function's graph must be zero.

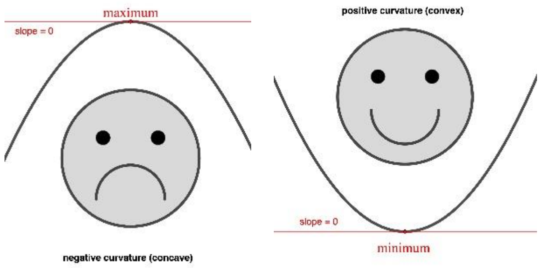


To determine the x-value of an extreme point, one can set the first derivative equal to zero and solve for x. The extreme value of y is then determined by substituting the value of x into the original function.

Example:

Calculate the extreme points of the function $f(x) = x^3 - 3x + 2$

1. We determine the derivative to obtain the gradient function: $f'(x) = 3x^2 - 3$
2. The gradient at an extreme point is zero, thus we need x for which: $3x^2 - 3 = 0 \Rightarrow x_1 = -1$ and $x_2 = 1$.
3. To determine the extreme values of $f(x)$, we determine $f(x_1)$ and $f(x_2)$: $f(-1) = (-1)^3 + 3 + 2 = 4$
   and $f(1) = 1^3 - 3 + 2 = 0$
4. The extreme values of the function are $(-1, 4)$ and $(1, 0)$.



In the above example it is not clear whether the extrema are maximum points or minimum points for the function.
To determine this, we need to understand the concept of curvature.

The **second derivative** of a function measures the **curvature** of the function.

A function with negative curvature has a maximum value and a positive curvature means that the function has minimum.

Example:

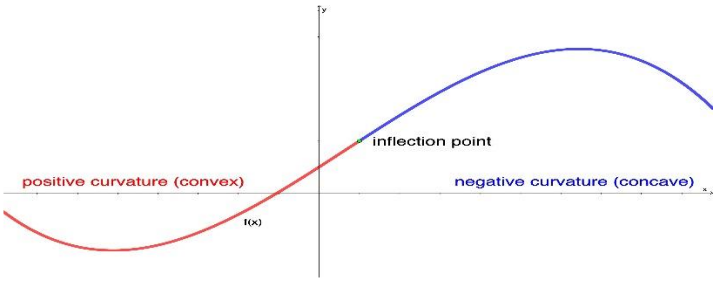
Determine whether the above extreme points, $(-1, 4)$ and $(1, 0)$, are maximum or minimum turning points.

1. Determine the second derivative for the curvature of the function: $f(x) = x^3 - 3x + 2 \Rightarrow f'(x) = 3x^2 - 3$
   $\Rightarrow f''(x) = 6x^1 = 6x$
2. Substitute the x-values of the extrema points into the second derivative, to determine whether the function is concave or convex at those points: $f''(-1) = 6 \cdot (-1) = -6 < 0$. The curvature is negative, so $(-1, 4)$ is a local maximum.
   $f''(1) = 6 \cdot 1 = 6 > 0$. The curvature is positive, so $(1, 0)$ is a local minimum.

A point $(x_0, f(x_0))$ with $f'(x_0) = 0$ and …
… $f''(x_0) > 0$ is a (local) minimum of the function $f(x)$.
… $f''(x_0) < 0$ is a (local) maximum of the function $f(x)$.
… $f''(x_0) = 0$ is a possible saddle point of the function $f(x)$.

Points of inflection

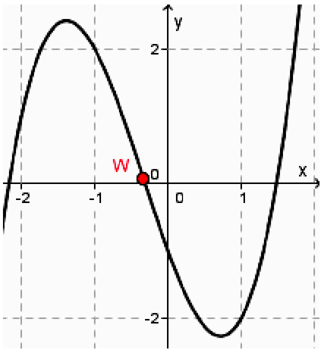
If the sign of a function's curvature changes at a point (i.e. the curve changes from convex to concave or vice versa), then that point is called a **point of inflection.**



$f''(x) = 0$ at a point of inflection, but not all occurrences of $f''(x) = 0$ are points of inflection. To determine whether a point is a point of inflection, we need the third derivative: if $f''(x_1) = 0$ and $f'''(x_1) \neq 0$, then $(x_1, f(x_1))$ is a point of inflection of $f(x)$.

Example:

Determine the point of inflection of $f(x) = x^3 + x^2 - 3x - 1$ and describe the curvature of the function.



We can estimate the position of the point of inflection (w) from the graph of the function.
To calculate it accurately, we determine the derivatives:
$f'(x) = 3x^2 + 2x - 3$ and $f''(x) = 6x + 2$ and $f'''(x) = 6$

By setting $f''(x) = 0$, we obtain all possible points of inflection:
$6x + 2 = 0 \Rightarrow x = -\frac{2}{6} = -\frac{1}{3}$

There might be a point of inflection at $x = -\frac{1}{3}$.

$f'''\left(-\frac{1}{3}\right) = 6 \neq 0 \Rightarrow$ there is a point of inflection at $x = -\frac{1}{3}$.

$f\left(-\frac{1}{3}\right) = (-\frac{1}{3})^3 + (-\frac{1}{3})^2 - 3 \cdot \left(-\frac{1}{3}\right) - 1 = \frac{2}{27}$.

Thus, the point of inflection is $(-\frac{1}{3}, \frac{2}{27})$.

To determine the curvature of the function, we select a point either to the right of the point of inflection or to the left of it. Let's consider $x = 0$, to the right of the point of inflection: $f''(0) = 6 \cdot 0 + 2 = 2 > 0$, so the function is positively curved to the right of $(-\frac{1}{3}, \frac{2}{27})$. Therefore, the curvature of the function is negative when $x < -\frac{1}{3}$ and positive when $x > -\frac{1}{3}$.

Determining a polynomial function based on a sample of points

In practice, we often don't know a function, but we do know some of its values. These might have been determined by measurement. Sometimes extrema and/or points of inflection are known or can, at least, be estimated. To be able to answer various questions, a mathematical model is sought. For simplicity, a polynomial function is often used.

Example:

$f(x)$ is a polynomial function of the 3rd degree. $f(1) = 2.5$. A zero occurs at $x = 0$. A (local) maximum occurs at $f(2) = 4$. Determine the function $f(x)$.

Example:

A polynomial function of the 3rd degree has the form $f(x) = ax^3 + bx^2 + cx + d$.

To be able to determine the four coefficients, we need (at least) four pieces of information (or four equations).

The derivative of the function is $f'(x) = 3ax^2 + 2bx + c$.

We can form a system of equations from the provided information:
$f(1) = 2.5 \quad \Rightarrow$ I: $a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = a + b + c + d = 2.5$
$f(0) = 0 \quad \Rightarrow$ II: $a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = \mathbf{d = 0}$
$f(2) = 4 \quad \Rightarrow$ III: $a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 8a + 4b + 2c + d = 4$
$f'(2) = 0 \quad \Rightarrow$ IV: $3a \cdot 2^2 + 2b \cdot 2 + c = 12a + 4b + c = 0$

Solving the system of linear equations, one obtains: $a = -0.5, b = 1, c = 2, d = 0$.

The function is, therefore, $f(x) = -0.5x^3 + x^2 + 2x$.

Example

Determine the equation of that polynomial of the 3rd degree which has a maximum at $(2, 1)$ and a point of inflection at $(0, -1)$.

Solution

$f(x) = -0.125\, x^3 + 1.5\, x - 1$

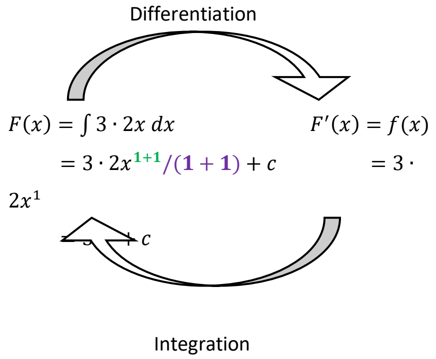
Integration

The anti-derivative (or indefinite integral)

A function $F(x)$ is **the anti-derivative** function of $f(x)$ if $F'(x) = f(x)$.
Determining the anti-derivative (also called the indefinite integral) is the inverse operation to finding a derivative.

$F(x)$ can also be written as $\int f(x)\, dx$. This notation corresponds with the "indefinite integral" nomenclature.

Example: Consider the anti-derivative of a power function $f(x) = 3 \cdot 2x$



To find the derivative of a power function: multiply by the exponent and then reduce the exponent by one.

Integration is the opposite of differentiation: increase the exponent by 1 and then divide by the new exponent.
c is the constant of integration. It should be clear that the derivative of $F(x)$ is unaffected by the value of c, so c could be any real value.

The result of integration is a new function. This new function is called the anti-derivative and is usually denoted with a capital letter, e.g. the anti-derivative of $f(x)$ is $F(x)$.

Integration of important functions:

- $\boldsymbol{f(x) = x^n \;\; (n \in \mathbb{Z}, n \neq -1)}$ $\Rightarrow \boldsymbol{F(x)} = \int x^n\, dx = \boldsymbol{\frac{x^{n+1}}{n+1} + c}$, e.g. if $f(x) = x^2$, then $F(x) = \frac{x^{2+1}}{2+1} + c = \frac{x^3}{3} + c$
- $\boldsymbol{f(x) = x^{-1} = 1/x}$ $\Rightarrow \boldsymbol{F(x)} = \int x^{-1}\, dx = \boldsymbol{\ln x + c}$
- $\boldsymbol{f(x) = e^x}$ $\Rightarrow \boldsymbol{F(x)} = \int e^x\, dx = \boldsymbol{e^x + c}$
- $\boldsymbol{f(x) = \sin(x)}$ $\Rightarrow \boldsymbol{F(x)} = \int \sin(x)\; dx = -\boldsymbol{\cos(x)} + c$
- $\boldsymbol{f(x) = \cos(x)}$ $\Rightarrow \boldsymbol{F(x)} = \int \cos(x)\; dx = \boldsymbol{\sin(x)} + c$

The integration and differentiation results for the above functions can be summarised in a table as follows:

indefinite integral:	$F(x)$	$\frac{x^{n+1}}{n+1} + c$	$\ln(x) + c$	$e^x + c$	$-\cos(x) + c$	$\sin(x) + c$
function:	$\boldsymbol{f(x)}$	$\boldsymbol{x^n}$	$\boldsymbol{1/x}$	$\boldsymbol{e^x}$	$\boldsymbol{\sin(x)}$	$\boldsymbol{\cos(x)}$
derivative:	$f'(x)$	$n \cdot x^{n-1}$	$-1/x^2$	e^x	$\cos(x)$	$-\sin(x)$

There are two useful integration rules:

Constant rule

$\boldsymbol{f(x) = k \cdot g(x) \Rightarrow F(x) = k \cdot G(x)}$
Example: $r(x) = 4x^3 \Rightarrow R(x) = 4 \cdot \frac{x^4}{4} + c = x^4 + c$

Sum rule

$\boldsymbol{f(x) = g(x) + h(x) \Rightarrow F(x) = G(x) + H(x)}$
Example: $s(x) = x^2 - 3x \Rightarrow S(x) = x^3/3 \;-\; 3x^2/2 \;+\; c$

Exercise:

A function l has the second derivative (function) $l'' = 6x - 6$. l has an extreme point at $(2, 5)$.

Determine the function l.

Solution:

$l = x^3 - 3x^2 + 9$

The definite integral

Theorem: (fundamental theorem of calculus)

If $f(x)$ is a continuous function defined on an interval $[a, b]$ (i.e. $f: [a, b] \rightarrow \mathbb{R}$), then:

$\boldsymbol{F(x) = \int_a^x f(t)\, dt}$ **is the anti-derivative of** $\boldsymbol{f(x)}$**, i.e.** $\boldsymbol{F'(x) = f(x)}$ … the first fundamental theorem of calculus

and

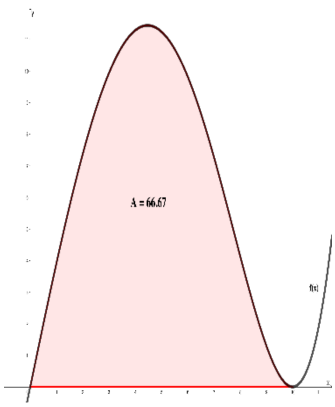
$$\int_a^b f(x)\, dx = F(x)\Big|_a^b = F(b) - F(a)$$

… the second fundamental theorem of calculus

$\int_a^b f(x)\, dx$ is called the **definite integral** (from a to b). This definite integral is equal to the value of the antiderivative at b (the upper limit) minus the anti-derivative at a (the lower limit).

Using integration to determine the area between a function curve and the horizontal axis

We wish to determine the area enclosed between the function's curve (from $x = 0$ to $x = 10$) and the horizontal axis.



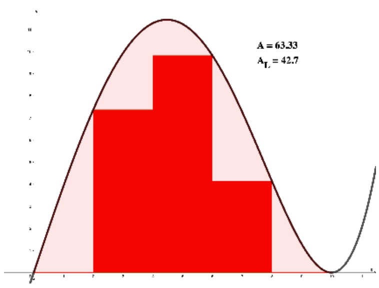
One way to approximate this area is to divide the surface into rectangles, determine the area of each rectangle and then add these areas together.

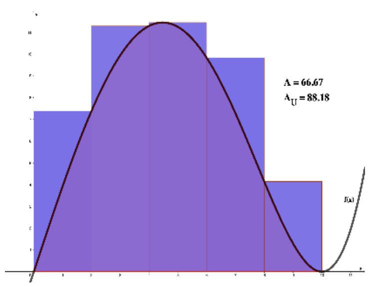
We could select rectangles that meet the curve from below (see the first figure below).

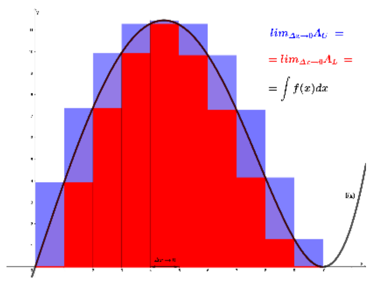
Alternatively, we could create rectangles that project beyond the curve (see the middle figure below).

The sum of the rectangle areas is too small for the first approach and too large for the second.

To determine a more precise value, we create very narrow rectangles. As in differential calculus, this results in the calculation of a limit, but this time it is the widths of the rectangles that tend toward zero.

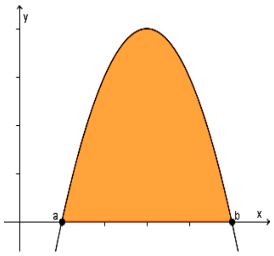


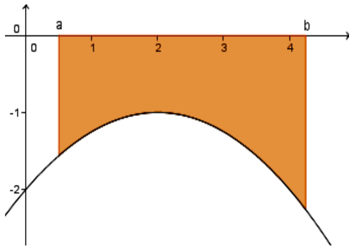


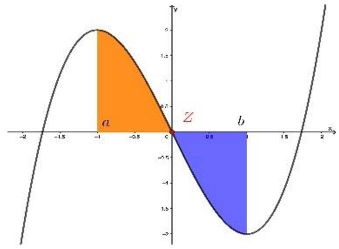


To approximate the area “under the function”, the areas of the (discrete) rectangles are summed; the notation for summation is Σ. To determine the area precisely for a continuous function, we determine a limit by integrating; the notation for integration is $\int$.

If a function $f(x)$ is defined on an interval $[a;b]$, then the area between the function’s curve, from a to b, and the horizontal axis is the **definite integral**:







$Area = \int_a^b f(x)\,dx$

$= F(b) - F(a)$

Absolute value of the area

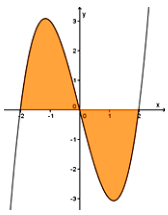
$= \left|\int_a^b f(x)\,dx\right| = |F(b) - F(a)|$

Absolute value of the area

$= \int_a^0 f(x)\,dx \;+\; \left|\int_0^b f(x)\,dx\right|$

$= F(0) - F(a) + |F(b) - F(0)|$

To determine the absolute value of the area, we must separate the integration whenever the function crosses the horizontal axis.



Example:

Calculate the absolute value of the area between the graph of y and the horizontal axis of $y = x^3 - 4x$.

To determine the zeros, we solve $y = x^3 - 4x = 0 \Rightarrow x \cdot (x^2 - 4) = 0 \Rightarrow x_1 = 0,\ \ x_2 = -2,\ \ x_3 = 2$.

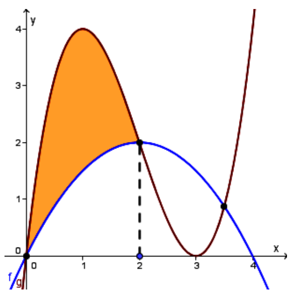
Determine the absolute area (of the shaded region): $\int_{-2}^{0} x^3 - 4x\,dx \;+\left|\int_0^2 x^3 - 4x\,dx\right|$

$$= \left(\frac{x^4}{4} - 2x^2\right)\Big|_{-2}^{0} + \left|\left(\frac{x^4}{4} - 2x^2\right)\Big|_{0}^{2}\right|$$

$$= \left(\frac{0^4}{4} - 2\cdot 0^2\right) - \left(\frac{(-2)^4}{4} - 2\cdot(-2)^2\right) + \left|\left(\frac{2^4}{4} - 2\cdot 2^2\right) - \left(\frac{0^4}{4} - 2\cdot 0^2\right)\right|$$

$$= 0 - (-4) + |(-4) - 0| = 4 + |-4| = 4 + 4 = 8$$

Using integration to determine the area between two function curves



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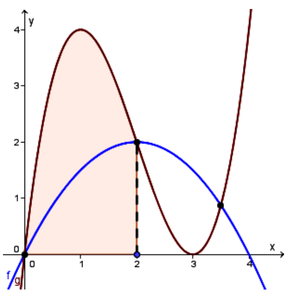

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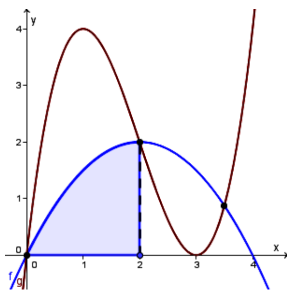
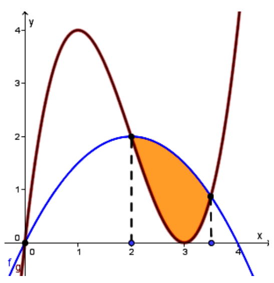
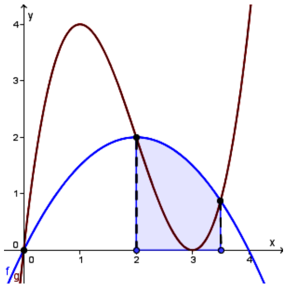
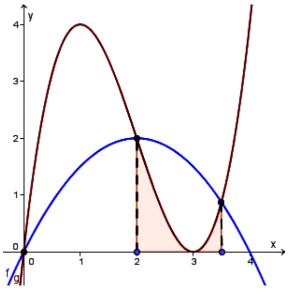


The yellow shaded area $= \int_0^2 g(x)\,dx - \int_0^2 f(x)\,dx$

 $=$  $-$ 

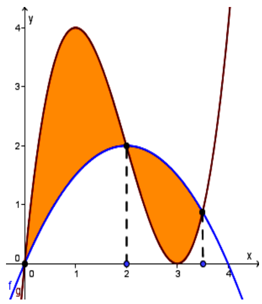
$$\textit{The yellow shaded area} = \int_2^{3.5} f(x)\,dx - \int_2^{3.5} g(x)\,dx$$

The area of the region between two functions f and g over the interval $[a;b]$ can be interpreted as the difference between the two regions lying between the graphs and the horizontal axis.

Example:

Calculate the surface enclosed between $f(x) = -0.5x^2 + 2x$ and $g(x) = x^3 - 6x^2 + 9x$ over the interval $[0,4]$.
Determine the points of intersection (for the integration limits): $-0.5x^2 + 2x = x^3 - 6x^2 + 9x$

$$\Rightarrow -x^3 + 5.5x^2 - 7x = 0$$
$$\Rightarrow x \cdot (-x^2 + 5.5x - 7) = 0$$
$$\Rightarrow x_1 = 0 \quad x_2 = 2 \quad x_3 = 3.5$$



Determine the absolute area: $A_1 + A_2$

where $A_1 = \int_0^2 g(x) - f(x)\,dx = \int_0^2 (x^3 - 6x^2 + 9x) - (-0.5x^2 + 2x)\,dx$

$$= \int_0^2 x^3 - 5.5x^2 + 7x\,dx$$
$$= \left(\frac{x^4}{4} - 5.5\frac{x^3}{3} + 3.5x^2\right)\Big|_0^2 = \frac{10}{3}$$

and $A_2 = \int_2^{3,5} f(x) - g(x)\,dx = \int_2^{3,5} -x^3 + 5.5x^2 - 7x\,dx = \left(-\frac{x^4}{4} + 5.5\frac{x^3}{3} - 3{,}5x^2\right)\Big|_2^{3,5} = 1.55$

The (absolute) area of the yellow-shaded region is $A_1 + A_2 = \frac{10}{3} + 1.55 = 4.88$ units.

Using integration to determine volume

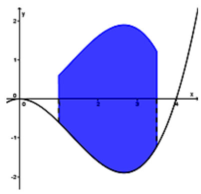
A surface can be defined by the region between a function's curve and the horizontal axis. If this surface is rotated about either of the two axes, a three-dimensional space is created. To calculate the volume of this space, we could decompose it into slices that rotate around the horizontal axis. The sum of the volumes of these slices provides an approximation of the volume we are looking for. If we make the slices very narrow (i.e. the widths tend toward zero), then the result becomes more precise.

Rotation about the horizontal axis

The volume of the space created by rotating a function's curve (from x_1 to x_2) about the horizontal axis is calculated as:

$V_x = \pi \cdot \int_{x_1}^{x_2} (f(x))^2\,dx = \pi \cdot \int_{x_1}^{x_2} y^2\,dx$ if $y = f(x)$

Example:



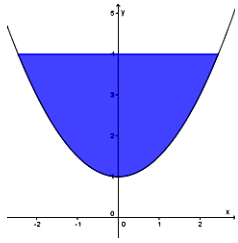
Calculate the volume of the space (or solid) created by rotating $f(x) = \frac{2}{10}x^3 - \frac{4}{5}x^2$ from $x_1 = 1$ to $x_2 = 3.5$ about the horizontal axis.

First, we determine $(f(x))^2$: $(f(x))^2 = \left(\frac{2}{10}x^3 - \frac{4}{5}x^2\right)^2 = \frac{1}{25}x^6 - \frac{8}{25}x^5 + \frac{16}{25}x^4$

The volume we seek is $V_x = \pi \cdot \int_1^{3.5} (f(x))^2 \, dx = \pi \cdot \int_1^{3.5} \frac{1}{25}x^6 - \frac{8}{25}x^5 + \frac{16}{25}x^4 \, dx = 18.44$ volume units

Rotation about the vertical axis

$V_y = \pi \cdot \int_{y_1}^{y_2} x^2 \ dy$



Example:

Calculate the volume of the space (or solid) created by rotating $y = 0.5x^2 + 1$ from $x_1 = 0$ to $x_2 = 1.22474$ about the vertical axis.

Determine the limits on the vertical axis: $x = 0 \Rightarrow y$ and $x = 1.22474 \Rightarrow y = 4$.
Determine x^2: $y = 0.5x^2 + 1 \Rightarrow y - 1 = 0.5x^2 \Rightarrow 2y - 2 = x^2$

Determine the volume: $V_y = \pi \cdot \int_1^4 x^2 \, dx = \pi \cdot \int_1^4 2y - 2 \, dy = 9$ volume units

Exercises:

1. Calculate the volume of a barrel obtained by rotating the graph of the function $g(x) = 25 - x^2/180$ about the horizontal axis
   over the interval $[-30\,;\ 30]$
2. A bowl is created by rotating the parabola $r(x) = x^2/4$ around the vertical axis.
   a) What is the bowl's volume if it is 1 cm deep at its deepest point?
   b) How deep is the liquid level (at the deepest point) if 5ml of oil is poured into the bowl?

Solutions:

1. $32800\,\pi \ \approx 103044.2$ volume units

a) $2\,\pi \approx 6.283$cm^3, b) $\sqrt{\frac{5}{2\,\pi}} \approx 0.892$ cm

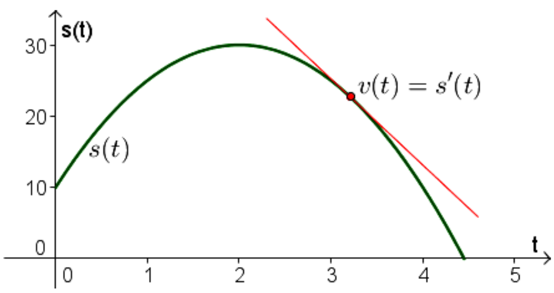
Motion calculations

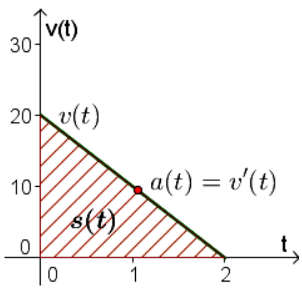
Another application of differential and integral calculus results from the relationship between distance and velocity. The derivative is a measure of the rate of change of one variable with respect to change in another variable. For example, velocity is the derivative of distance with respect to time, as it measures the rate of change of distance over time:

$v(t) = s'(t)$ and $s(t) = \int v(t)\,dt$
where $s(t)$ represents the distance (e.g. in metres, m) travelled up to time t
and $v(t)$ represents the velocity (e.g. in metres per second, m/s) at time t.

The derivative of velocity measures the rate of change of velocity over time, i.e. acceleration.

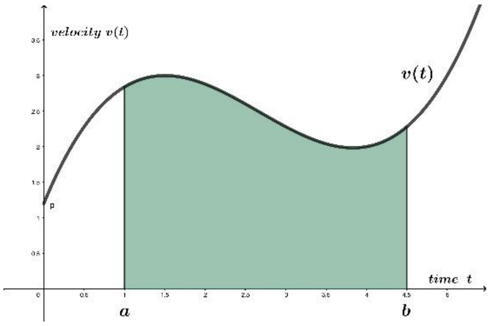
$a(t) = v'(t) = s''(t)$ and $v(t) = \int a(t)\,dt$
where $a(t)$ represents acceleration (e.g. in metres per second squared, m/s^2)





Exercise:

A velocity function $v(t)$ is shown in the following graph:



What does the area of the green-shaded region represent?

Solution:

The green-shaded region has an area equal to the integral of $v(t)$ over the period $t = 1$ to $t = 4.5$. Thus, the area of the green-shaded region represents the distance travelled by the object in the specific interval of 3.5 seconds.

Costs, revenue and profit

Cost function

Costs are incurred in the production of a good. These costs are made up of fixed costs and variable costs. Those costs which do not vary with the quantity produced are called fixed costs. Variable costs are those costs which depend directly on the produced quantity for a good.

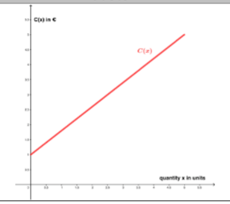
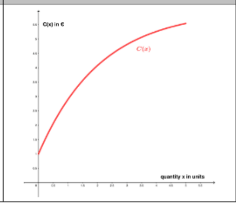
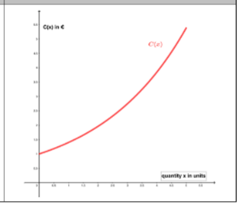
The total cost of production is fixed costs + variable costs: $C(x) = C_f(x) + C_v(x)$
where $C(x)$ represents total costs, $C_f(x)$ represents the fixed costs and $C_v(x)$ represents the variable costs associated with producing x units of a good.

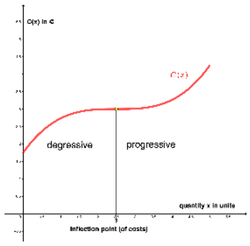
Examples of fixed costs:	rent for the business premises, salaries
Examples of variable costs:	raw materials, transport costs

Marginal cost function

Marginal cost is the incremental (or additional) cost of producing one additional unit of the good.
The marginal cost function is $C'(x)$, i.e. marginal cost is calculated as the first derivative of the cost function.
A cost function may have different forms:

- The total costs might increase in proportion to the quantity produced, i.e. they are a linear function of the produced quantity.
- Degressive costs are characterised by the total costs growing proportionately more slowly than the number of units (e.g. because of more efficient processes). The graph of degressive costs has a negative curvature, i.e. the total cost function is concave.
- Progressive costs are total costs that grow relatively faster than the number of units produced (e.g. due to higher wear and tear of the machines, overtime payments, etc.). Progressive costs have a graph with a positive curvature (i.e. the curve is convex).

proportional (or linear) costs	degressive costs	progessive costs
		



Some cost functions change from concave to convex (or vice versa) as the quantity produced increases, i.e. the cost function has a point at which $C''(x) = 0$ (a point of inflection) – see the diagram above on the right.

Average cost function

The average costs per unit produced are $\bar{C}(x) = \frac{C(x)}{x}$.

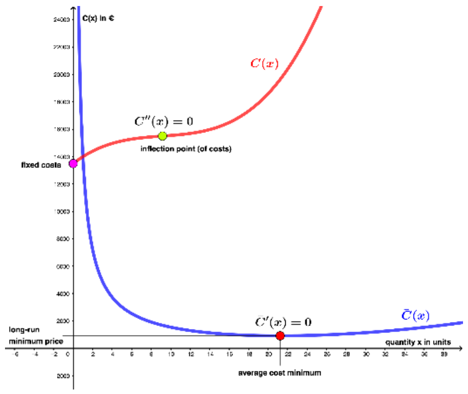
The minimum average cost occurs at the quantity x_{MinAC} for which $\bar{C}'(x_{MinAC}) = 0$ and $\bar{C}''(x_{MinAC}) > 0$. The minimum average cost per unit is $\bar{C}(x_{MinAC})$.

Long-term minimum price

Over the long term, the price per unit that a producer charges must cover at least the lowest possible average cost per unit, i.e. the long-term minimum price is $\bar{C}(x_{MinAC})$.

Example:

At which production quantity is the average unit cost per unit minimised if the total cost function is $C(x) = 2x^3 - 55x^2 + 555x + 13500$?



$C(x) = 2x^3 - 55x^2 + 555x + 13500$

The average cost function is $\bar{C}(x) = \frac{2x^3-55x^2+555x+13500}{x}$

$= 2x^2 - 55x + 555 + 13500/x$

The average cost per unit is (potentially) minimised when

$\bar{C}'(x) = 4x - 55 - 13500/x^2 = 0$

$\Rightarrow \quad 4x^3 - 55x^2 - 13500 = 0$

$\Rightarrow \quad x = 21.23$

$\bar{C}''(x) = 4 + 2 \cdot 13500/x^3$

$\Rightarrow \bar{C}''(21.23) = 4 + 2 \cdot 13500/21.23^3 = 4{,}0002 > 0$, so the average cost per unit is **minimised** at a quantity of 21.23 units.

Exercise:

For each of the situations below, determine the quadratic total cost function, the output quantity at which average costs are minimised and the lowest price that the producer should charge.

a) The fixed costs are €250, the total cost for 100 units is €760 and for 500 units it is €3000.

b) For 10 units the total cost is €860, for 20 units it is €940 and for 30 units it is €1040.

c) The fixed costs are €1000. At 400 units the total cost is €25000 and the marginal cost is €100.

Solution:

a) $C(x) = 0.001x^2 + 5x + 250$
Average cost per unit is minimised at a production quantity of 500 units.
The lowest price the producer should charge is €6 per unit.

b) $C(x) = 0.1x^2 + 5x + 800$
Average cost per unit is minimised at a production quantity of 89.44 units.
The lowest price the producer should charge is €22.89 per unit.

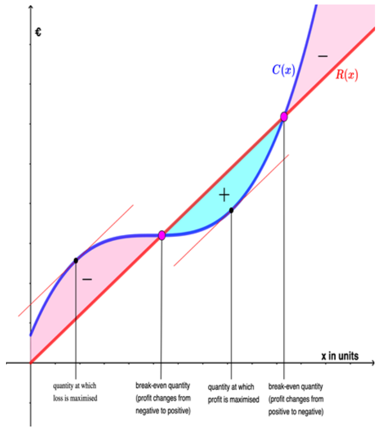
c) $(x) = 0.1x^2 + 20x + 1000$
Average cost per unit is minimised at a production quantity of 100 units.
The lowest price the producer should charge is €40 per unit.

Revenue and profit

A monopolistic producer can charge whatever price it wants for its product. However, it must consider that demand decreases with higher prices. The price function (as a function of x, the demanded quantity) is represented by $p(x)$.

The **maximum price** that the producer can charge is $p(0)$. At this price, consumers are no longer willing to buy the good.

The market is said to be **saturated** at the quantity x at which $p(x) = 0$, i.e. the producer cannot sell more than quantity x, as it would have to give away the good for free.



The total revenue of a producer is price (per unit) times number of units demanded (or sold), so the **revenue function** is $R(x) = p(x) \cdot x$.

Profit is total revenue minus total costs. Thus, the **profit function** is $\Pi(x) = R(x) - C(x)$.

The producer is profitable when the revenue is greater than the total costs. Usually, when the quantity sold is very low, the revenues do not cover the total costs, so the profit is negative, i.e. the producer makes a (positive) loss.

Break-even point

A **break-even point** is any point at which profit is zero, $\Pi(x) = R(x) - C(x) = 0$, i.e. when total revenue equals total costs, $R(x) = C(x)$.

The break-even quantities are the zeros of the profit function.

Profit-maximisation

An obvious quantity of interest is the quantity at which profit is maximised.

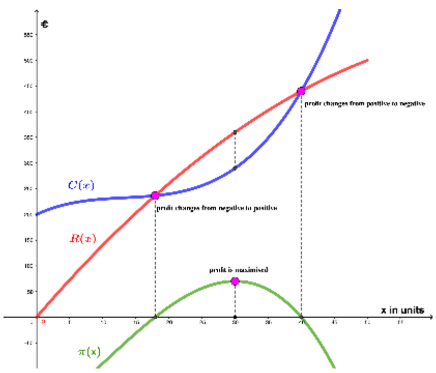
Profit is maximised at the quantity at which the slope of the profit function is zero and its curvature is negative (i.e. the profit curve is concave). Thus, profit is maximised when $\Pi'(x) = 0$ and $\Pi''(x) < 0$.

Example:

A company's cost function is $C(x) = 0.01x^3 - 0.4x^2 + 6x + 200$ and the price function is $p(x) = -0.1x + 15$.

Calculate the maximum possible profit and the quantity limits for the company's range of profitability.

$$\Pi(x) = R(x) - C(x) = p(x) \cdot x - C(x) = (-0.1x + 15) \cdot x - (0.01x^3 - 0.4x^2 + 6x + 200) = -0.01x^3 + 0.3x^2 + 9x - 200$$



Maximum profit

$\Pi'(x) = -0.03x^2 + 0.6x + 9 = 0 \quad \Rightarrow x = 30$
$\Pi''(x) = -0.06x + 0.6$, so $\Pi''(30) = -0.06 \cdot 30 + 0.6x = -1{,}2 < 0$,
so profit is maximised at a quantity of 30 units.
The maximum profit is $\Pi'(x) = 70$, i.e. €70.

The range of profitability

Break-even occurs when $\Pi(x) = -0.01x^3 + 0.3x^2 + 9x - 200 = 0 \Rightarrow\ x_1 = 18$ and $x_2 = 40$.
The company will make a (non-negative) profit (i.e. it will be profitable) at all quantities from 18 units to 40 units.

Exercise:

1) Determine the linear demand function (price function) and the revenue function, the maximum price and the saturation quantity if the following is true: at a price of €40 per unit the company sells 100 units; at €20 it sells 200 units.
2) Determine the quadratic demand function and the revenue function, the maximum price and the saturation quantity for the following demand function values: at a price of €400 per unit demand is 100 units; at €160 demand is 300 units; at €70 demand is 400 units.
3) The following is known for a quadratic cost function: fixed costs are €400, the average cost per unit is minimised at a quantity of 200 units and the lowest price to be charged is €11 per unit. The demand function is $p(x) = 28 - 0.04x.$
   Determine the total cost function, the price and quantity when profit is maximised and the maximum profit.

Solution:

1) $p(x) = -0.2x + 60$ and $R(x) = -0.2x^2 + 60x$
   The maximum price is €60.
   The saturation quantity is 300 units.
2) $p(x) = 0.001x^2 - 1.6x + 550$ and $R(x) = 0.001x^3 - 1.6x^3 + 550x$
   The maximum price is €550.
   The saturation quantity is 500 units.
3) $C(x) = 0.01x^2 + 7x + 400$
   Profit is maximised at a quantity of 210 units and a price of €19.60 per unit.
   The maximum profit is €1805.

Descriptive statistics

Descriptive statistics concerns the collection, presentation and analysis of **data**.

The **elements of interest** in an investigation or an analysis are the objects that we want to know more about. For example, we might be interested in voters in a country, companies in a particular sector or students in a class.

Characteristics (or variables) are the features of interest for the elements. For example, we might be interested in the preferred political party of voters, companies' share prices at the end of last month or the grades of students in a class. Variables whose values are expressed as numbers are called **quantitative variables**. Variables with values expressed some other way (e.g. in words) are called **qualitative variables**.

Data is a list of all the observed (or recorded) values of the characteristics for the elements one wishes to describe. Each observation in the data represents an element.

Population: the set of all elements which one wishes to describe.

Sample: those elements for which one has the data – this is a subset of the population. The sample could be the population, but usually it is only a small part of the population. The aim of a (small) sample is to allow us to draw conclusions (or inferences) about the population. It is, therefore, important that the sample is **representative** of the population.

The simplest form of collected data is a list of the observed values of the variables. This list is called the **raw data**. If we represent a variable as X and we have n observations in the data set, then the observed values of the variable are denoted by x_1, x_n, x_n, ..., x_n.

The raw data can be summarised as a table showing the absolute and relative frequencies of the observed values. The **absolute frequency** of a variable value measures how often this value occurs in the data. The **relative frequency** of the variable value measures the proportion of observations that have that value.

A frequency distribution may be represented as a table or as a graph (or chart), e.g. a pie chart, a bar chart, a histogram, etc.

Measures of central tendency

When analysing data, it is useful to calculate metrics that provide us with a feel for the data. The first type of metric we consider are the measures of central tendency. These describe, roughly, where most of the data values tend to lie.

Arithmetic mean

The arithmetic mean of a data sample for a variable X is denoted by $\bar{x}$ and is calculated as $\bar{x} = \frac{x_1+x_2+\ldots+x_n}{n}$.

Mode

The mode is the most frequently occurring value in the data set, i.e. it is the value with the highest frequency.

Median

The median of a data sample for a variable X is determined as follows:

- arrange the observed values x_1, x_2, x_3, ..., x_n from smallest to largest.
- for an odd number of observations, the median is the value of the middle observation.
- for an even number of observations, the median is the arithmetic mean of the two values in the middle.

Although the measures of central tendency often provide useful information about the distribution of a sample of data, care must be taken when interpreting their values.

The arithmetic mean can be calculated only for quantitative variables. However, even here, it is not always appropriate. For example, it makes no sense to calculate the arithmetic mean of the house numbers on a street.

The mode can always be determined, but it is not always very informative. For example, the mode of the data set
1, 1, 1, 1000, 1000
is 1, but this result is not useful as a summary of the true distribution of the values.
The mode is particularly appropriate when one value occurs much more frequently in the data set than any other values.

The median can be determined only for variables that can be meaningfully ordered. However, it too is not always very informative. The median of the data sample
1, 1, 1, 1000, 1000
is 1. Again, this does not provide an accurate reflection of the average of the values.

Outliers

An outlier is an extreme value (very high or very low) that has a low relative frequency in the data. For example, in the data set 50, 27, 62, 1000, 35, 2, 42
the fourth observation (with the value 1000) is a high outlier and the sixth observation (with the value 2) is a low outlier.

A weakness of the arithmetic mean, compared to the other measures of central tendency, is that it is sensitive to outliers. For example, the arithmetic mean of the values
1, 2, 2, 5
is 2.5. If a further value of 1000 is added to the data, then the arithmetic mean is 202. The arithmetic mean has been **distorted** upward by the high outlier and no longer provides a useful measure of where most of the values lie.

The mode usually does not change significantly when outliers are added. For example, the mode of the data sample
1, 2, 2, 3, 4
is 2 and this does not change when the outlier 1000 is added.

Similarly, the median is also **robust against outliers**. The median of the list
1, 2, 2, 3, 4
is equal to 2. If the outlier 1000 is added, then the new median is $\frac{2+3}{2} = 2.5$, so only slightly greater than before.

Boxplot

The median splits the sample into two halves: equal numbers of observations lie below and above the median. Quartiles further split the data observations, as the name suggests, into four regions with each subset containing one quarter of the observations.

Quartiles

The quartiles q_1, q_2, q_3 of a sample are determined as follows:

- the observation values x_1, x_n, x_n, ..., x_n are arranged by size, from smallest to largest.
- the median is the second quartile q_2.
- the first quartile q_1 is the median of the values lying up to the median.
- the third quartile q_3 is the median of the values that lie above the median.

Range

The range is the difference between the largest value (maximum) and the smallest value (minimum) of the sample.

A boxplot is a graphical representation of the quartiles and the range.

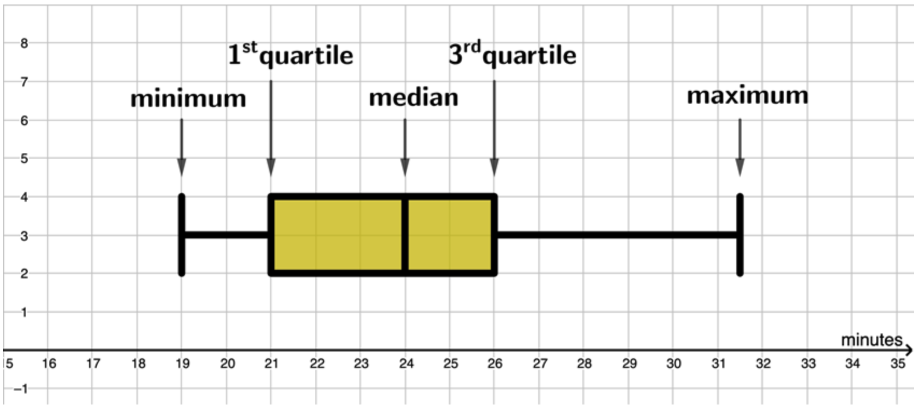
Minimum: smallest value of the sample

Lower quartiles: q_1

Median: $\tilde{x} = q_2$

Upper quartiles: q_3

Maximum: largest value of the sample



Measures of dispersion

One might argue that a statistician with one hand in the freezer and the other hand on a hot stovetop should, on average, feel fine. This example demonstrates that the arithmetic mean does not provide a measure of the variability of the values in a data sample.

To understand variability, we need to look at how the values deviate about the mean.

The deviation about the mean for the i^{th} observation value of the variable X is $x_1 - \overline{x}$. If we add up the deviations for all observation values and divide by n, the number of observations, then we would have the average deviation about the mean for the sample. However, some of the deviations about the mean are positive and some are negative. Mathematically, the sum of the deviations will be zero for any data sample. Thus, the average deviation about the mean is useless as a measure of dispersion. To get around this issue, statisticians use the squared deviation about the mean, $(x_i - \overline{x})^2$, for analysis purposes.

Variance

The **population variance** is the average squared deviation about the mean for the population. The population variance for a variable X is denoted by ${\sigma_X}^2$ and is calculated as

${\sigma_X}^2 = \frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+\ldots+(x_N-\overline{x})^2}{N}$ where N is the number of objects in the population.

The **sample variance** for a variable X is denoted by ${s_X}^2$ and is calculated as

${s_X}^2 = \frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+\ldots+(x_n-\overline{x})^2}{n-1}$ where n is the number of objects in the sample. This sample variance is an unbiased estimator of the true population variance.

The variance is measured in a different unit to the variable itself, e.g. if the variable is measured in metres, then variance of the variable is measured in square metres. This makes the interpretation of the variance a bit difficult.

Standard deviation

For interpretation purposes, one usually uses the square root of the variance, which is called the standard deviation.

For a variable X, the **population standard deviation** is $\sigma_X = \sqrt{{\sigma_X}^2}$ and the **sample standard deviation** is $s_X = \sqrt{{s_X}^2}$. The sample standard deviation s_X is an unbiased estimator of σ_X, the true standard deviation for the population.

Probability theory

Experiments

Chance plays a key role in many processes. We cannot predict exactly what the weather will be like in two months' time. Too many factors, which are not known today, influence future weather patterns. Nor is it possible to determine in advance which numbers will be drawn in the lottery. For such processes, we define an **experiment** as a process with a random outcome for the variable of interest. Each repetition of the experiment is called a **trial**.

Examples of experiments:

1. A die is thrown and the number of dots on the upper side are observed. "Number of dots" is the variable.
2. A driver approaches an intersection with traffic lights. What colour does the traffic light show? The variable is "colour of the traffic lights".
3. A ball is rolled in a game of roulette. At which number does the ball come to rest?
4. A doctor examines a patient. What is the patient's blood group?
5. A commuter arrives at a tram station. How long does it take for the next tram to arrive?

An experiment can be repeated a number of times under the same conditions. Such repetitions are called trials. For each trial, it is clear what the possible outcomes are for the variable of interest, but it is not possible to say with certainty which of these will occur.

The possible results of a trial of an experiment are expressed as a **set of possible outcomes Ω**.

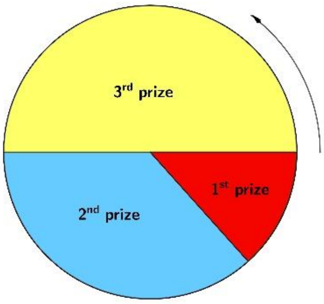
Probability of an event

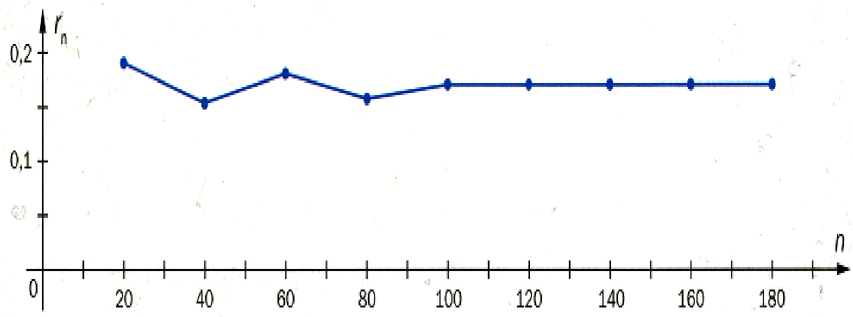
The aim of probability calculation is to calculate the probability of any event A occurring. But what is "probability"?

Probability as relative frequency

What is the probability of winning the first prize on a spin of the tombola wheel shown below?

One way of determining the probability is to spin the wheel again and again, and to record how often the first prize comes up. From this absolute frequency we can determine the relative frequency.





The more trials we perform for the experiment, the closer the observed relative frequency lies to a particular value. This concept is generally referred to as the empirical law of large numbers, which dates to Jacques Bernoulli (1654–1705). We interpret the probability of an event as its relative frequency when the random experiment is repeated very often. The more trials we perform for the experiment, the more accurate the observed relative frequency (as an estimate for the true probability).

In the tombola example, the variable is "type of prize that is spun". The possible values for the outcome of a trial of the experiment are 1, 2 and 3. Hence, the sample space is $\Omega = \{1, 2, 3\}$.
An event, like the sample space, is written as a set, e.g. the event that the first prize or the third prize is thrown is $\{1, 3\}$. We might represent the event that the first prize is thrown as A. Then $A = \{1\}$. The probability that event A will occur is written as $P(A)$. In our example we estimate that $P(A) \approx 0.17 = 17\ \%$.

The complementary event of A is the event that A does not occur. It is denoted as A'. The probability that A does not occur is $P(A') = 1 - P(A)$.

Classical probability

If a random experiment has a finite number of outcomes and each of these has the same probability, then the probability of an event A occurring is calculated as $P(A) = \frac{\text{Number of favourable outcomes for event } A}{\text{Number of possible outcomes for teh experiment}} = \frac{|A|}{|\Omega|}$
where $|A|$ and $|\Omega|$ denote the number of elements in the event A, respectively the number of elements in the sample space Ω.
This approach is called the classical probability approach or the Laplace probability approach. It applies only when the outcomes of a trial are all equally likely.

Exercise:

Sally rolls three fair dice. Harry bets that the Sally will roll a total of eight dots.
What is the probability that Harry will win the bet?
What is the probability that Harry will lose the bet?

Solution:

Let (i, j, k) represent a possible outcome of the experiment where i represents the number of dots on the first die, j represents the number of dots on the second die and k represents the number of dots on the third die. i, j and k can each have the values 1, 2, 3, 4, 5 or 6.

There are $6^3 = 216$ possible outcomes for this experiment. Each of these outcomes has the same probability.

Let B be the event that Harry wins the bet. The favourable outcomes for B are:
(1,1,6), (1,6,1), (6,1,1), (1,2,5), (1,5,2), (2,1,5), (2,5,1), (5,1,2), (5,2,1), (1,3,4), (1,4,3), (3,1,4), (3,4,1), (4,1,3), (4,3,1)
There are 15 favourable outcomes for B, so $P(B) = \frac{15}{216} = 0.069 = 6.9\%$. Harry will win the bet with a probability of 6.9%.
Harry will lose the bet with a probability of $P(B') = 1 - \frac{15}{216} = 0.931 = 93.1\%$.

Binomial distribution

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$, e.g. $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. $n!$ is read as "n factorial".

There are $\frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{k! \cdot (n-k)!}$ ways of selecting k objects from n objects if the order doesn't matter and repetitions are not allowed. This number of selections can be represented as $\binom{n}{k}$, read as "n combination k".

Example:

How many ways are there of choosing four seats out of ten free seats in a theatre (the order does not matter)?

There are $\binom{10}{4} = \frac{10!}{4! \cdot 6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1) \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} = 210$ possible selections.

The binomial distribution is a commonly used distribution. It may be applied whenever the following four conditions are met:

1) A random experiment is repeated a number of times.
2) For every trial of the experiment there are two outcomes. We usually refer to these outcomes as "success" and "failure".
3) The trials are independent of one other.
4) p, the probability of success, remains the same for each trial.

Example:

It is known that one person out of a thousand experiences an adverse reaction to a vaccination.

A total of 2000 people are vaccinated. Calculate the probability that exactly two people will experience an adverse reaction.

Review of the criteria:

1) The experiment is repeated 2000 times.
2) Each trial has two possible outcomes: a) there is an adverse reaction or b) there is no adverse reaction.
3) One person's reaction to the vaccination is not affected by another person's reaction, i.e. the outcomes of the trials are independent of one another.
4) The probability that an adverse reaction will occur is $p = \frac{1}{1000} = 0.001$ for each person.

The criteria are fulfilled, the binomial distribution may be applied.

Calculating probabilities for the binomial distribution

If a random experiment with only two possible outcomes (success or failure) for each trial is repeated n times
and the trials are all independent of one another
and the probability of success p is the same for every trial,
then the number of successes X is binomially distributed; this is written as $X \sim B(n, p)$.

The binomial distribution has two parameters. Given n and p we can determine any probability for X.

The probability that there will be k successes is $\boldsymbol{P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}}$.

The expected value of X is $\boldsymbol{\mu_X = n \cdot p}$ and the standard deviation of X is $\boldsymbol{\sigma_X = \sqrt{n \cdot p \cdot (1-p)}}$

Example continued:

Each of the $n = 2000$ vaccinations is a trial of the random experiment. For each trial, the probability of an adverse reaction is $p = 0.001$.

The random variable here is A, the number of of people with adverse reactions: $A \sim B(2000, 0.001)$.

Therefore, the probability that exactly two people will experience an adverse reaction is

$$P(A = 2) = \binom{2000}{2} \cdot 0.001^2 \cdot (1 - 0.001)^{2000-2} = \binom{2000}{2} \cdot 0.001^2 \cdot 0.999^{1998} \approx 0.2708 = 27.08\%.$$

Example:

For the vaccination example, determine the probability that three to five people will have an adverse reaction.

$A \sim B(2000, 0.001)$

Three to five people have an adverse reaction means that either three or four or five people have an adverse reaction.

Therefore, $P(3 \le A \le 5) = P(A = 3) + P(A = 4) + P(A = 5)$

$$= \binom{2000}{3} \cdot 0.001^3 \cdot (1 - 0.001)^{1997} + \binom{2000}{4} \cdot 0.001^4 \cdot (1 - 0.001)^{1996} + \binom{2000}{5} \cdot 0.001^5 \cdot (1 - 0.001)^{1995}$$

$$\approx 0.1805 + 0.0902 + 0.0361 = 0.3068 \approx 30.68\%$$

The probability that three to five people will have an adverse reaction is approximately 30.7%.

Exercise:

Calculate the probability of getting a "6" three times when throwing ten fair dice.

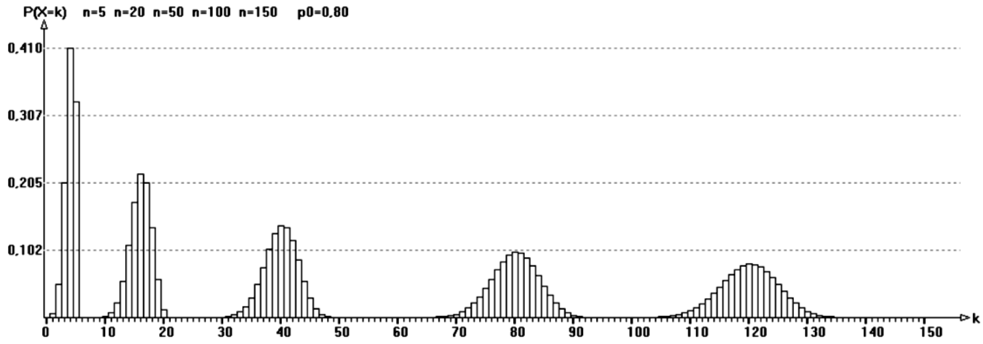
Solution:

If S is the number of "6"s, then the required probability is $P(S = 3) = \binom{10}{3} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(1 - \frac{1}{6}\right)^{10-3} = 0.1550 = 15.50\%$

Normal distribution

Gaussian bell-shaped curve

The following graphs demonstrate how the histogram of a binomial distribution changes as the number of trials increases, but the probability of success (for each trial) remains the same.



It is interesting to note that, as we increase the number of trials n, …

… the expected value shifts further to the right.
… more bars become visible.
… the heights of the bars in the histogram reduce.
… the histogram becomes wider, i.e. the average deviation about the mean increases.
… the histogram appears smoother (i.e. more "continuous").

As n becomes very large, the values on the vertical axis tend toward the following function: $f_N(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot \left(\frac{k-\mu}{\sigma}\right)^2}$

The graph of this (probability density) function is called a Gaussian bell curve.

Normal distribution

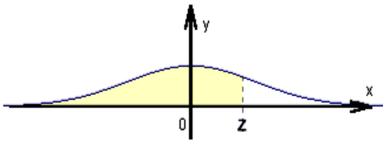
A variable X whose values x follow a distribution described by the (probability density) function $f_N(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot \left(\frac{x-\mu}{\sigma}\right)^2}$ is called the normal distribution with the parameters μ and σ.
μ is the expected value of the variable and σ is the standard deviation of the variable.
The graph of the function $f_N(x)$ is a symmetrical bell-shaped curve, centred on the expected value μ and with a standard deviation of σ.

To determine probabilities for ranges of x, we determine the area under the bell-shaped curve for the relevant interval of x-values. Determining these areas is not easy, so it would be useful to have a table of values for the probabilities. Most statistical textbooks provide tables of probability values for the normal distribution with the parameters $\mu = 0$ and $\sigma = 1$. This is a special case of the normal distribution, called the **standard normal distribution**, usually we denote a variable with the standard normal distribution as Z and we write $f_Z(x) = \varphi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot z^2}$ for the probability density function.

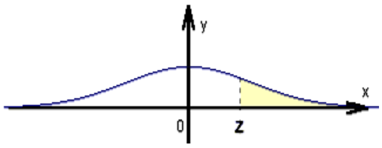
The cumulative distribution function of Z is the antiderivative of $\varphi(z)$ and is denoted by $P(Z \leq z) = \Phi(z)$. The tables in textbooks provide the values for $\Phi(z)$.

Any variable X with a normal distribution can be transformed into a variable Z with the standard normal distribution by the **standardisation** $\boldsymbol{Z} = \frac{X-\mu_X}{\sigma_X}$.

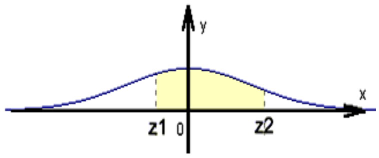
The probability that $Z \leq z$ is:
$P(Z \leq z) = \Phi(z)$



The probability that $Z > z$ is:
$P(Z \geq z) = 1 - P(Z \leq z)$
$= 1 - \Phi(z) = \Phi(-z)$



The probability that Z lies between z_1 and zx_2 is:
$P(z_1 \leq Z \leq z_2) = \Phi(z_2) - \Phi(z_1)$



Example:

In pin production, the pin length L is normally distributed with $\mu = 20$mm and $\sigma = 1.2$mm.
A randomly selected pin is measured.
What is the probability that the pin is shorter than 19mm?

The required probability is $P(L \leq 19) = P\left(\frac{L-20}{1.2} \leq \frac{19-20}{1.2}\right) = P(Z \leq -0.833) = \phi(-0.833) = 0.20327 \approx 20.33\%$. The last result is read from the table in a textbook.

What is the probability that the pin is longer than 22mm?

The required probability is $P(L > 22) = 1 - P(L \leq 22) = 1 - P\left(\frac{L-20}{1.2} \leq \frac{22-20}{1.2}\right)$

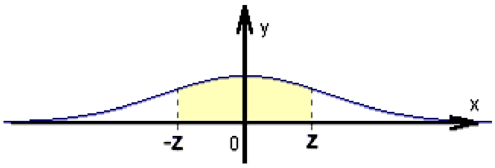
$$= 1 - P(Z \leq 1.667)$$
$$= 1 - \phi(1.667) = 1 - 0.95154 = 0.04846 \approx 4.85\%.$$

What is the probability that the pin is between 21mm and 22mm long?

The required probability is $P(21 \leq L \leq 22) = P(L \leq 22)$ $= P(L \leq 22) - P(L \leq 21)$

$$= 0.95154 - P\left(\frac{L-20}{1.2} \leq \frac{21-20}{1.2}\right)$$
$$= 0.95154 - P(Z \leq 0.833)$$
$$= 0.95154 - \phi(0.833)$$
$$= 0.95154 - 0.79673 = 0.15481 \approx 15.48\%$$

For a symmetrical interval from z standard deviations below μ to z standard deviations above μ, the probability is $P(\mu - z \cdot \sigma \ \leq X \leq \mu + z \cdot \sigma) = \Phi(z) - \Phi(-z) = \Phi(z) - \big(1 - \Phi(z)\big) = 2\Phi(z) - 1$.



Example:

In which symmetrical range about the expected value do the lengths of 90% of the pins lie?

We require that $0.05 \leq Z \leq 0.95$.

$\Phi(z_{Lower}) = 0.05 \Rightarrow z_{Lower} = -1.64$ and $\Phi\big(z_{Upper}\big) = 0.95 \Rightarrow z_{Upper} = 1.64$ … from tables in a statistics textbook.

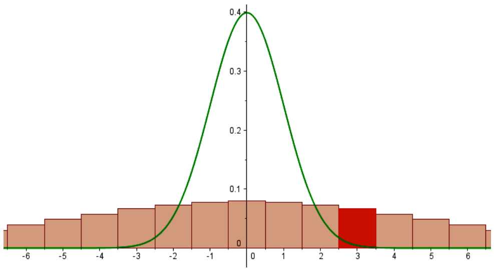
$z_{Lower} = \frac{x_{Lower} - \mu}{\sigma} = -1.64 \Rightarrow x_{Lower} - \mu = -1.64 \cdot \sigma \Rightarrow x_{Lower} = -1.64 \cdot \sigma + \mu = -1.64 \cdot 1.2 + 20 = 18.032$
and $z_{Upper} = \frac{x_{Upper} - \mu}{\sigma} = 1.64 \Rightarrow x_{Upper} - \mu = 1.64 \cdot \sigma \Rightarrow x_{Upper} = 1.64 \cdot \sigma + \mu = 1.64 \cdot 1.2 + 20 = 21.968$

90% of all pins have lengths between 18.032mm and 21.968mm.

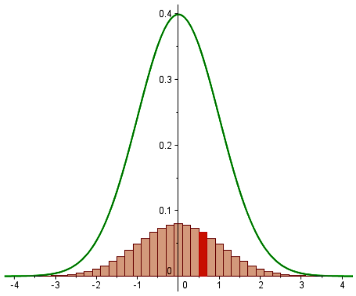
The normal distribution as an approximation of the binomial distribution

The areas of the rectangles in the histogram of the binomial distribution represent probabilities. If we add up the areas of all the rectangles, the result is 1, i.e. an aggregate probability of 100%.

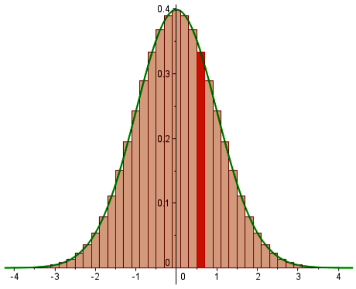
1) The highest frequency of the binomial distribution occurs not at 0, but at the expected value μ.
If we **shift the histogram μ units to the left**, then it is centred about 0.
However, the diagram demonstrates that the histogram of the binomial distribution is "wider" than the Gaussian bell curve.



2) To compress the binomial distribution horizontally, we **divide all the horizontal-axis values by σ**. This has the effect of reducing the rectangle widths and the distribution now fits under the Gaussian bell-shaped curve.
However, the histogram is still too compressed vertically compared to the Gaussian bell curve.



3) Since the rectangular areas of the histogram must add up to 1, their heights must be multiplied by σ. The result is an almost smooth, symmetrical bell-shaped curve.



Note, that in the above manipulations we have:
1. subtracted the expected value from the variable, and
2. divided by the variable's standard deviation.

Thus, we have carried out a standardisation of the binomial variable.

From the above discussion we conclude that, if n is very large, the binomial distribution can be approximated by a normal distribution with expected value $\mu = n \cdot p$ and standard deviation $\sigma = \sqrt{n \cdot p \cdot (1-p)}$.

Specifically, a binomial distribution may be approximated by a normal distribution if $\sigma = \sqrt{n \cdot p \cdot (1-p)}$

III Business administration

Lead author (adapter): Barbara Waldhauser

Chapter 1 – The foundations of business

Stephen Skripak, Anastasia Cortes, and Anita Walz

Learning objectives

1. Describe the concept of stakeholders and identify the stakeholder groups relevant to an organization 2. Discuss and be able to apply the macro business environment model to an industry or emerging technology 3. Explain other key terms related to this chapter including entrepreneur; profit; revenue.

Why is Apple successful?

In 1976 Steve Jobs and Steve Wozniak created their first computer, the Apple I.[1] They invested a mere $1,300 and set up business in Jobs' garage. Three decades later, their business—Apple Inc.—has become one of the world's most influential and successful companies. Jobs and Wozniak were successful entrepreneurs: those who take the risks and reap the rewards associated with starting a new business enterprise. Did you ever wonder why Apple flourished while so many other young companies failed? How did it grow from a garage start-up to a company generating over $233 billion in sales in 2015? How was it able to transform itself from a nearly bankrupt firm to a multinational corporation with locations all around the world? You might conclude that it was the company's products, such as the Apple I and II, the Macintosh, or more recently its wildly popular iPod, iPhone, and iPad. Or, you could decide that it was its dedicated employees, management's wiliness to take calculated risks, or just plain luck – that Apple simply was in the right place at the right time.

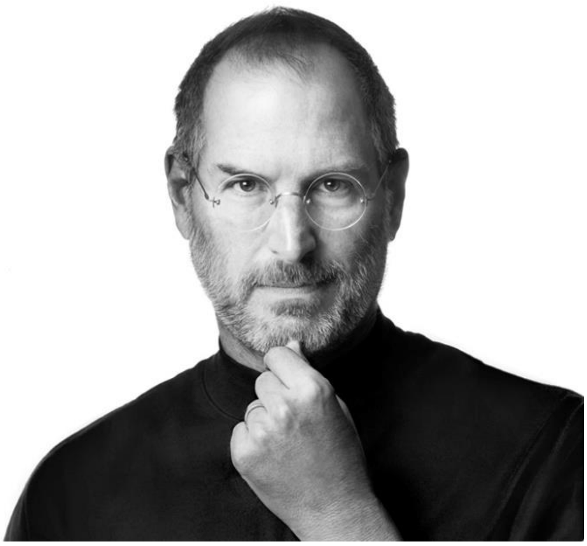


Figure 1: Steve Jobs

Before we draw any conclusions about what made Apple what it is today and what will propel it into a successful future, you might like to learn more about Steve Jobs, the company's cofounder and former CEO. Jobs was instrumental in the original design of the Apple I and, after being ousted from his position with the company, returned to save the firm from destruction and lead it onto its current path. Growing up, Jobs had an interest in computers. He attended lectures at Hewlett-Packard after school and worked for the company during the summer months. He took a job at Atari after graduating from high school and saved his money to make a pilgrimage to India in search of spiritual enlightenment. Following his India trip, he attended Steve Wozniak's "Homebrew Computer Club" meetings, where the idea for building a personal computer surfaced.[2] "Many colleagues describe Jobs as a brilliant man who could be a great motivator and positively charming. At the same time his drive for perfection was so strong that employees who did not meet his demands [were] faced with blistering verbal attacks."[3] Not everyone at Apple appreciated Jobs' brilliance and ability to motivate. Nor did they all go along with his willingness to do whatever it took to produce an innovative, attractive, high-quality product. So, at age thirty, Jobs found himself ousted from Apple by John Sculley, whom Jobs

[1] This vignette is based on an honors thesis written by Danielle M. Testa, "Apple, Inc.: An Analysis of the Firm's Tumultuous History, in Conjunction with the Abounding Future" (Lehigh University), November 18, 2007.
[2] Lee Angelelli (1994). "Steve Paul Jobs." Retrieved from: http://ei.cs.vt.edu/~history/Jobs.html
[3] Ibid.

himself had hired as president of the company several years earlier. It seems that Sculley wanted to cut costs and thought it would be easier to do so without Jobs around. Jobs sold $20 million of his stock and went on a two-month vacation to figure out what he would do for the rest of his life. His solution: start a new personal computer company called NextStep. In 1993, he was invited back to Apple (a good thing, because neither his new company nor Apple was doing well).

Steve Jobs was definitely not known for humility, but he was a visionary and had a right to be proud of his accomplishments. Some have commented that "Apple's most successful days occurred with Steve Jobs at the helm."[4]

Jobs did what many successful CEOs and managers do: he learned, adjusted, and improvised.[5] Perhaps the most important statement that can be made about him is this: he never gave up on the company that once turned its back on him.(e) Steve Jobs, or (f) some combination of these options.

Introduction

As the story of Apple suggests, today is an interesting time to study business. Advances in technology are bringing rapid changes in the ways we produce and deliver goods and services. The Internet and other improvements in communication (such as smartphones, video conferencing, and social networking) now affect the way we do business. Companies are expanding international operations, and the workforce is more diverse than ever. Corporations are being held responsible for the behavior of their executives, and more people share the opinion that companies should be good corporate citizens. Because of the role they played in the worst financial crisis since the Great Depression, businesses today face increasing scrutiny and negative public sentiment.[6]

Economic turmoil that began in the housing and mortgage industries as a result of troubled subprime mortgages quickly spread to the rest of the economy. In 2008, credit markets froze up and banks stopped making loans. Lawmakers tried to get money flowing again by passing a $700 billion Wall Street bailout, now-cautious banks became reluctant to extend credit. Without money or credit, consumer confidence in the economy dropped and consumers cut back on spending. Unemployment rose as troubled companies shed the most jobs in five years, and 760,000 Americans marched to the unemployment lines.[7] The stock market reacted to the financial crisis and its stock prices dropped by 44 percent while millions of Americans watched in shock as their savings and retirement accounts took a nose dive. In fall 2008, even Apple, a company that had enjoyed strong sales growth over the past five years, began to cut production of its popular iPhone. Without jobs or cash, consumers would no longer flock to Apple's fancy retail stores or buy a prized iPhone.[8] Since then, things have turned around for Apple, which continues to report blockbuster sales and profits. But not all companies or individuals are doing so well. The economy is still struggling, unemployment is high (particularly for those ages 16 to 24), and home prices have not fully rebounded from the crisis.

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[4] Cyrus Farivar (2006). "Apple's first 30 years; three decades of contributions to the computerIndustry." Macworld, June 2006, p. 2.
[5] Dan Barkin (2006). "He made the iPod: How Steve Jobs of Apple created the new millennium's signature invention." Knight Ridder Tribune Business News, December 3, 2006, p. 1.
[6] Jon Hilsenrath, Serena Ng, and Damian Paletta (2008). "Worst Crisis Since '30s, With No End Yet in Sight," Wall Street Journal, Markets, September 18, 2008. Retrieved from: http://www.wsj.com/articles/SB122169431617549947
[7] Steve Hargreaves (2008). "How the Economy Stole the Election," CNN.com. Retrieved from: http://money.cnn.com/galleries/2008/news/0810/gallery.economy_election/index.html
[8] Dan Gallagher (2008). "Analyst says Apple is cutting back production as economy weakens." MarketWatch. Retrieved from: http://www.marketwatch.com/story/apple-cutting-back-iphone-production-analyst-says?amp%3Bdist=msr_1

As you go through the course with the aid of this text, you'll explore the exciting world of business. We'll introduce you to the various activities in which business people engage—accounting, finance, information technology, management, marketing, and operations. We'll help you understand the roles that these activities play in an organization, and we'll show you how they work together. We hope that by exposing you to the things that businesspeople do, we'll help you decide whether business is right for you and, if so, what areas of business you'd like to study further.

Getting down to business

A business is any activity that provides goods or services to consumers for the purpose of making a profit. Be careful not to confuse the terms revenue and profit. Revenue represents the funds an enterprise receives in exchange for its goods or services. Profit is what's left (hopefully) after all the bills are paid. When Steve Jobs and Steve Wozniak launched the Apple I, they created Apple Computer in Jobs' family garage in the hope of making a profit. Before we go on, let's make a couple of important distinctions concerning the terms in our definitions. First, whereas Apple produces and sells goods (Mac, iPhone, iPod, iPad, Apple Watch), many businesses provide services. Your bank is a service company, as is your Internet provider. Hotels, airlines, law firms, movie theaters, and hospitals are also service companies. Many companies provide both goods and services. For example, your local car dealership sells goods (cars) and also provides services (automobile repairs). Second, some organizations are not set up to make profits. Many are established to provide social or educational services. Such not-for profit (or nonprofit), organizations include the United Way of America, Habitat for Humanity, the Boys and Girls Clubs, the Sierra Club, the American Red Cross, and many colleges and universities. Most of these organizations, however, function in much the same way as a business. They establish goals and work to meet them in an effective, efficient manner. Thus, most of the business principles introduced in this text also apply to nonprofits.

Business participants and activities

Let's begin our discussion of business by identifying the main participants of business and the functions that most businesses perform. Then we'll finish this section by discussing the external factors that influence a business' activities.

Participants

Every business must have one or more owners whose primary role is to invest money in the business. When a business is being started, it's generally the owners who polish the business idea and bring together the resources (money and people) needed to turn the idea into a business. The owners also hire employees to work for the company and help it reach its goals. Owners and employees depend on a third group of participants— customers. Ultimately, the goal of any business is to satisfy the needs of its customers in order to generate a profit for the owners.

Stakeholders

Consider your favorite restaurant. It may be an outlet or franchise of a national chain (more on franchises in a later chapter) or a local "mom and pop" without affiliation to a larger entity. Whether national or local, every business has stakeholders – those with a legitimate interest in the success or failure of the business and the policies it adopts. Stakeholders include customers, vendors, employees, landlords, bankers, and others (see Figure 2). All have a keen interest in how the business operates, in most cases for obvious reasons. If the business fails, employees will need new jobs, vendors will need new customers, and banks may have to write off loans they made to the business. Stakeholders do not always see things the same way – their interests sometimes conflict with each other. For example, lenders are more likely to appreciate high profit margins that ensure the loans they made will be repaid, while customers would probably appreciate the lowest possible prices. Pleasing stakeholders can be a real balancing act for any company.

Figure 2: Business Stakeholders

Functional areas of business

The activities needed to operate a business can be divided into a number of functional areas. Examples include management, operations, marketing, accounting, and finance. Let's briefly explore each of these areas.

Management

Managers are responsible for the work performance of other people. Management involves planning for, organizing, leading, and controlling a company's resources so that it can achieve its goals. Managers plan by setting goals and developing strategies for achieving them. They organize activities and resources to ensure that company goals are met and staff the organization with qualified employees and managers lead them to accomplish organizational goals. Finally, managers design controls for assessing the success of plans and decisions and take corrective action when needed.

Operations

All companies must convert resources (labor, materials, money, information, and so forth) into goods or services. Some companies, such as Apple, convert resources into tangible products—Macs, iPhones, etc. Others, such as hospitals, convert resources into intangible products — e.g., health care. The person who designs and oversees the transformation of resources into goods or services is called an operations manager. This individual is also responsible for ensuring that products are of high quality.

Marketing

Marketing consists of everything that a company does to identify customers' needs (i.e. market research) and design products to meet those needs. Marketers develop the benefits and features of products, including price and quality. They also decide on the best method of delivering products and the best means of promoting them to attract and keep customers. They manage relationships with customers and make them aware of the organization's desire and ability to satisfy their needs.

Accounting

Managers need accurate, relevant and timely financial information, which is provided by accountants. Accountants measure, summarize, and communicate financial and managerial information and advise other managers on financial matters. There are two fields of accounting. Financial accountants prepare financial statements to help users, both inside and outside the organization, assess the financial strength of the company. Managerial accountants prepare information, such as reports on the cost of materials used in the production process, for internal use only.

Finance

Finance involves planning for, obtaining, and managing a company's funds. Financial managers address such questions as the following: How much money does the company need? How and where will it get the necessary money? How and when will it pay the money back? What investments should be made in plant and equipment? How much should be spent on research and development? Good financial management is particularly important when a company is first formed, because new business owners usually need to borrow money to get started.

External forces that influence business activities

Apple and other businesses don't operate in a vacuum; they're influenced by a number of external factors. These include the economy, government, consumer trends, technological developments, public pressure to act as good corporate citizens, and other factors. Collectively, these forces constitute what is known as the "macro environment" – essentially the big picture world external to a company over which the business exerts very little if any control. Figure 3 "Business and Its Environment" sums up the relationship between a business and the outside forces that influence its activities.

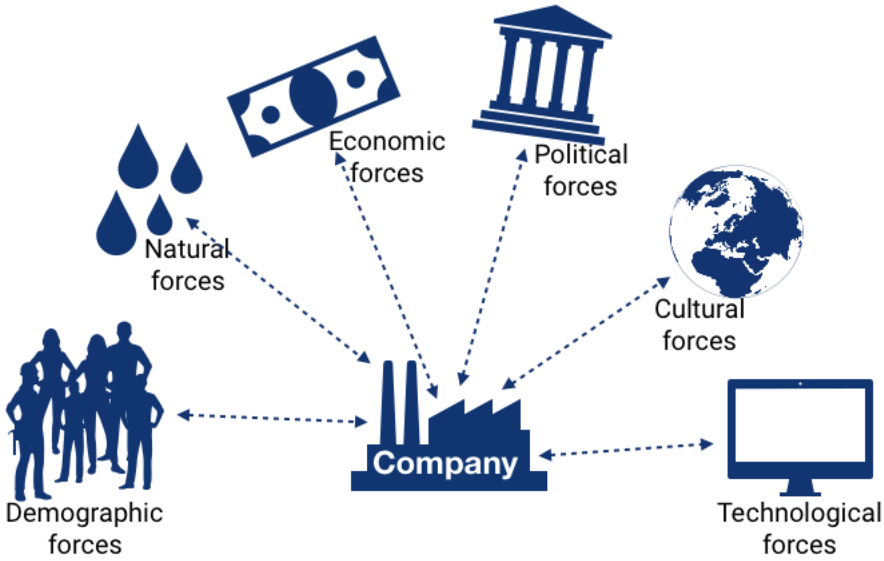

Figure 3: The business and its environment

One industry that's clearly affected by all these factors is the fast-food industry. Companies such as Taco Bell, McDonald's, Cook-Out and others all compete in this industry. A strong economy means people have more money to eat out. Food standards are monitored by a government agency. Preferences for certain types of foods are influenced by consumer trends (fast food companies are being pressured to make their menus healthier). Finally, a number of decisions made by the industry result from its desire to be a good corporate citizen. For example, several fast-food chains have responded to environmental concerns by eliminating Styrofoam containers.[9]

Of course, all industries are impacted by external factors, not just the food industry. As people have become more conscious of the environment, they have begun to choose new technologies, like all-electric cars to replace those that burn fossil fuels. Both established companies, like Nissan with its Nissan Leaf, and brand-new companies like Tesla have entered the market for all-electric vehicles. While the market is still small, it is expected to grow at a compound annual growth rate of 19.2% between 2013 and 2019.10

PESTEL analysis

One useful tool for analyzing the external environment in which an industry or company operates is the PESTEL model. PESTEL is an acronym, with each of the letters representing an aspect of the macro-environment that a business needs to consider in its planning. Let's briefly run through the meaning of each letter.

P stands for the political environment. Governments influence the environment in which businesses operate in many ways, including taxation, tariffs, trade agreements, labor regulations, and environmental regulations.

E represents the economic environment. As we will see in detail in a later chapter, whether the economy is growing or not is a major concern to business. Numerous economic indicators have been created for the specific purpose of measuring the health of the economy.

S indicates the sociocultural environment, which is a category that captures societal attitudes, trends in national demographics, and even fashion trends. The term demographics applies to

[9] David Baron (2003). "Facing-Off in Public." Stanford Business. August 2003, pp. 20-24. Retrieved from: https://www.gsb.stanford.edu/sites/gsb/files/2003August.pdf

any attribute that can be used to describe people, such as age, income level, gender, race, and so on. As a society's attitudes or its demographics change, the market for goods and services can shift right along with it.

T is for technological factors. In the last several decades, perhaps no force has impacted business more than the emergence of the internet. Nearly instantaneous access to information, e-commerce, social media, and even the ability to control physical devices from remote locations have all come about due to technological forces.

The second E stands for environmental forces, which in this case means natural resources, pollution levels, recycling, etc. While the attitudes of a society towards the natural environment would be considered a sociocultural force, the level of pollution, the supply of oil, etc. would be grouped under this second E for environment.

Finally, the L represents legal factors. These forces often coincide with the political factors already discussed, because it is politicians (i.e., government) that enacts laws. However, there are other legal factors that can impact businesses as well, such as decisions made by courts that may have broad implications beyond the case being decided.

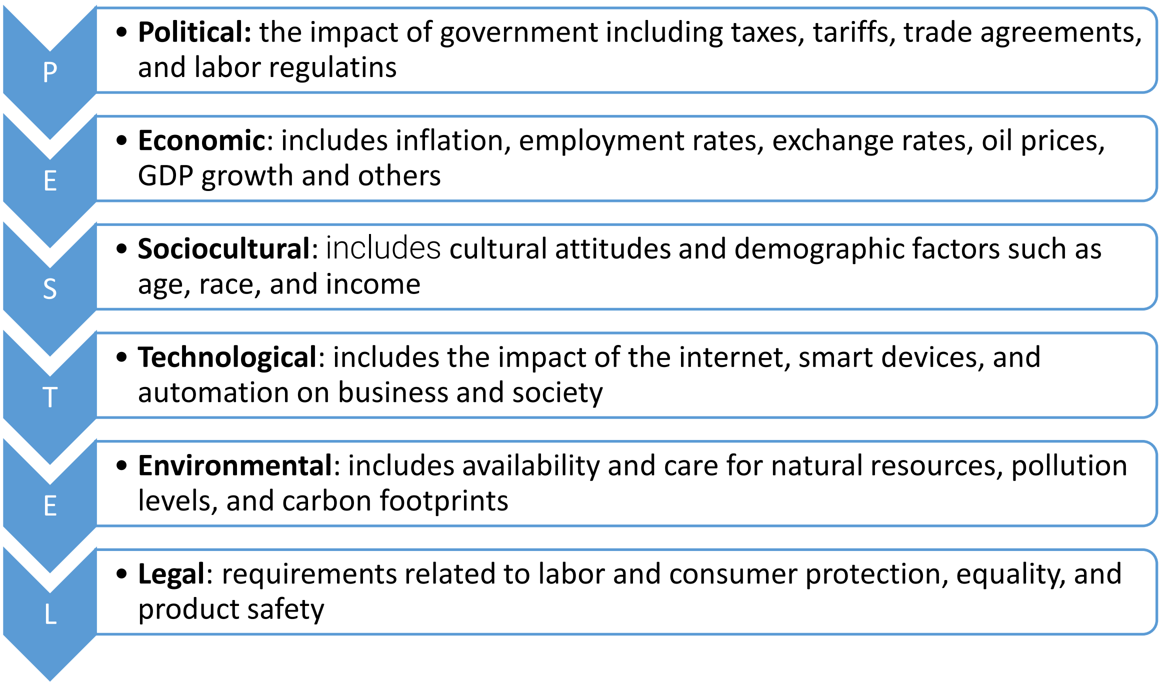

Figure 4: Business and its Environment – PESTEL

When conducting PESTEL analysis, it is important to remember that there can be considerable overlap from category to category. It's more important that businesses use the model to thoroughly assess its external environment, and much less important that they get all the forces covered under the "right" category. It is also important to remember that an individual force, in itself, is not inherently positive or negative but rather presents either an opportunity or a threat to different businesses. For example, societal attitudes moving in favor of green energy are an opportunity for those with capabilities in wind, solar, and other renewables, while presenting a threat, or at least a need to change, to companies whose business models depend exclusively on fossil fuels.

Key takeaways

The main participants in a business are its owners, employees, and customers.

Every business must consider its stakeholders, and their sometimes conflicting interests, when making decisions.

The activities needed to run a business can be divided into functional. The business functions correspond fairly closely to many majors found within a typical college of business.

Businesses are influenced by such external factors as the economy, government, and other forces external to the business. The PESTEL model is a useful tool for analyzing these forces.

Chapter 2 – Economics and business

Stephen Skripak, Anastasia Cortes, and Anita Walz

Learning objectives

1. Describe the foundational philosophies of capitalism and socialism.
2. Discuss private property rights and why they are key to economic development.
3. Discuss the concept of GDP (gross domestic product).
4. Explain the difference between fiscal and monetary policy.
5. Discuss the concept of the unemployment rate measurement.
6. Discuss the concepts of inflation and deflation.
7. Explain other key terms related to this chapter including supply; demand; equilibrium price; monopoly; recession; depression.

What is economics?

To appreciate how a business functions, we need to know something about the economic environment in which it operates. We begin with a definition of economics and a discussion of the resources used to produce goods and services.

Resources: Inputs and outputs

Economics is the study of the production, distribution, and consumption of goods and services. Resources are the inputs used to produce outputs. Resources may include any or all of the following:

- Land and other natural resources
- Labor (physical and mental)
- Capital, including buildings and equipment
- Entrepreneurship
- Knowledge

Resources are combined to produce goods and services. Land and natural resources provide the needed raw materials. Labor transforms raw materials into goods and services. Capital (equipment, buildings, vehicles, cash, and so forth) are needed for the production process. Entrepreneurship provides the skill, drive and creativity needed to bring the other resources together to produce a good or service to be sold to the marketplace.

Because a business uses resources to produce things, we also call these resources factors of production. The factors of production used to produce a shirt would include the following:

- The land that the shirt factory sits on, the electricity used to run the plant, and the raw cotton from which the shirts are made
- The laborers who make the shirts
- The factory and equipment used in the manufacturing process, as well as the money needed to operate the factory
- The entrepreneurship skills and production knowledge used to coordinate the other resources to make the shirts and distribute them to the marketplace

Input and output markets

Many of the factors of production are provided to businesses by households. For example, households provide businesses with labor (as workers), land and buildings (as landlords), and capital (as investors). In turn, businesses pay households for these resources by providing them with income, such as wages, rent, and interest. The resources obtained from households are then used by businesses to produce goods and services, which are sold to provide businesses with revenue. The revenue obtained by businesses is then used to buy additional resources, and the cycle continues. This is described in Figure 5: The Circular Flow of Inputs and Outputs, which illustrates the dual roles of households and businesses:

- Households not only provide factors of production (or resources) but also consume goods and services.
- Businesses not only buy resources but also produce and sell both goods and services

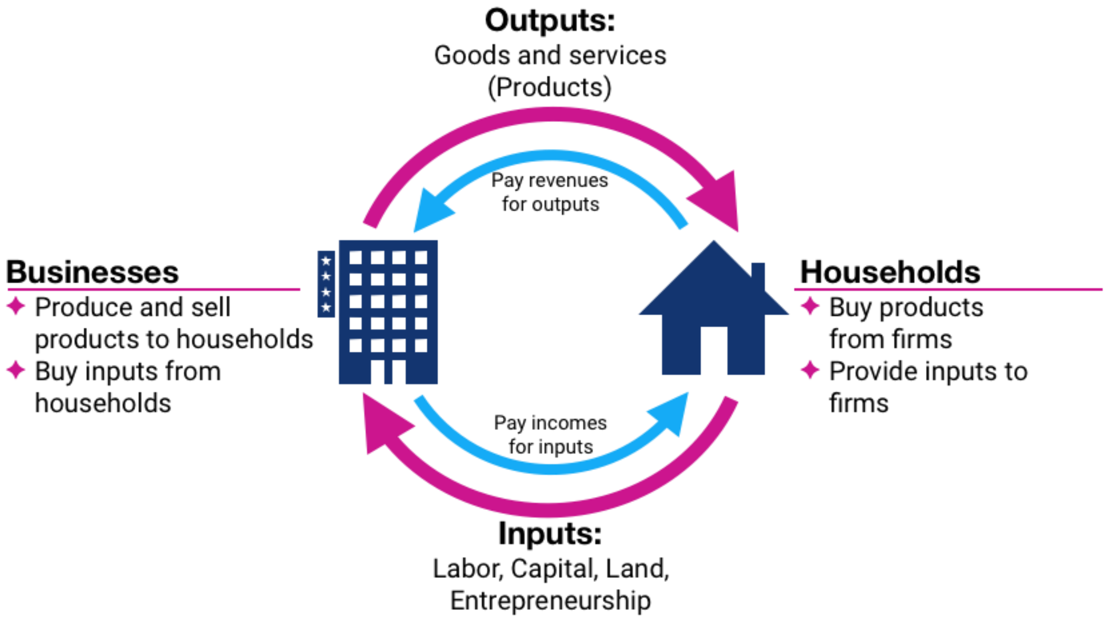

Figure 5: The Circular Flow of Inputs and Outputs

Economic systems

Economists study the interactions between households and businesses and look at the ways in which the factors of production are combined to produce the goods and services that people need. Basically, economists try to answer three sets of questions:

- What goods and services should be produced to meet consumers' needs? In what quantity? When?
- How should goods and services be produced? Who should produce them, and what resources, including technology, should be combined to produce them?
- Who should receive the goods and services produced? How should they be allocated among consumers?

The answers to these questions depend on a country's economic system—the means by which a society (households, businesses, and government) makes decisions about allocating resources to produce products and about distributing those products. The degree to which individuals and business owners, as opposed to the government, enjoy freedom in making these decisions varies according to the type of economic system.

Generally speaking, economic systems can be divided into two systems: planned systems and free market systems.

Planned systems

In a planned system, the government exerts control over the allocation and distribution of all or some goods and services. The system with the highest level of government control is communism. In theory, a communist economy is one in which the government owns all or most enterprises. Central planning by the government dictates which goods or services are produced, how they are produced, and who will receive them. In practice, pure communism is practically nonexistent today, and only a few countries (notably North Korea and Cuba) operate under rigid, centrally planned economic systems.

Under socialism, industries that provide essential services, such as utilities, banking, and health care, may be government owned. Some businesses may also be owned privately. Central planning allocates the goods and services produced by government-run industries and tries to ensure that the resulting wealth is distributed equally. In contrast, privately owned companies are operated for the purpose of making a profit for their owners. In general, workers in socialist economies work fewer hours, have longer vacations, and receive more health care, education, and child-care benefits than do workers in capitalist economies. To offset the high cost of public services, taxes are generally steep. Examples of countries that lean towards a socialistic approach include Venezuela, Sweden, and France.

Free market system

The economic system in which most businesses are owned and operated by individuals is the free market system, also known as capitalism. In a free market economy, competition dictates how goods and services will be allocated. Business is conducted with more limited government involvement concentrated on regulations that dictate how businesses are permitted to operate. A key aspect of a free market system is the concept of private property rights, which means that business owners can expect to own their land, buildings, machines, etc., and keep the majority of their profits, except for taxes. The profit incentive is a key driver of any free market system. The economies of the United States and other countries, such as Japan, are based on capitalism. However, a purely capitalistic economy is as rare as one that is purely communist. Imagine if a service such as police protection, one provided by government in the United States, were instead allocated based on market forces. The ability to pay would then become a key determinant in who received these services, an outcome that few in American society would consider to be acceptable.

How economic systems compare

In comparing economic systems, it can be helpful to think of a continuum with communism at one end and pure capitalism at the other, as in Figure 3.2 on the next page. As you move from left to right, the amount of government control over business diminishes. So, too, does the level of social services, such as health care, child-care services, social security, and unemployment benefits. Moving from left to right, taxes are correspondingly lower as well.

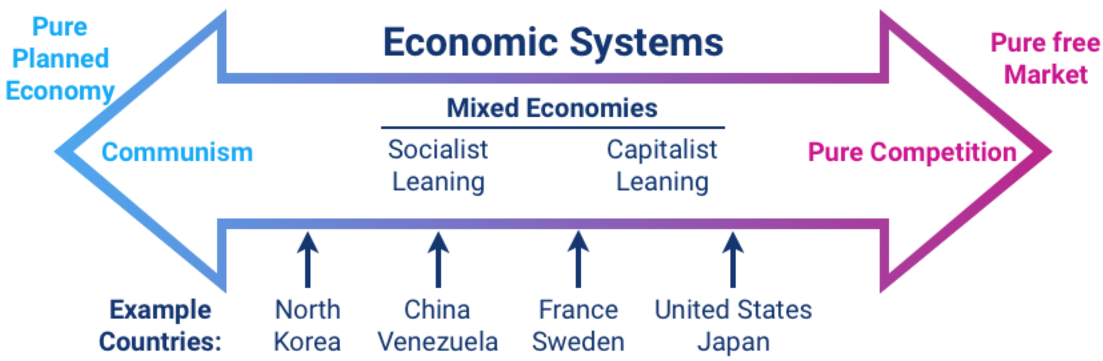

Figure 6: The Economic Spectrum

Mixed market economies

Though it's possible to have a pure communist system, or a pure capitalist (free market) system, in reality many economic systems are mixed. A mixed market economy relies on both markets and the government to allocate resources. In practice, most economies are mixed, with a leaning towards either free market or socialistic principles, rather than being purely one or the other. Some previously communist economies, such as those of Eastern Europe and China, are becoming more mixed as they adopt more capitalistic characteristics and convert businesses previously owned by the government to private ownership through a process called privatization. By contrast, Venezuela is a country that has moved increasingly towards socialism, taking control of industries such as oil and media through a process called nationalization.

The U.S. economic system

Like most countries, the United States features a mixed market system: though the U.S. economic system is primarily a free market system, the federal government controls some basic services, such as the postal service and air traffic control. The U.S. economy also has some characteristics of a socialist system, such as providing social security retirement benefits to retired workers.

The free market system was espoused by Adam Smith in his book The Wealth of Nations, published in 1776. According to Smith, competition alone would ensure that consumers received the best products at the best prices. In the kind of competition he assumed, a seller who tries to charge more for his product than other sellers would not be able to find any buyers. A job-seeker who asks more than the going wage won't be hired. Because the "invisible hand" of competition will make the market work effectively, there won't be a need to regulate prices or wages. Almost immediately, however, a tension developed among free market theorists between the principle of laissez-faire—leaving things alone—and government intervention. Today, it's common for the U.S. government to intervene in the operation of the economic system. For example, government exerts influence on the food and pharmaceutical industries through the Food and Drug Administration, which protects consumers by preventing unsafe or mislabeled products from reaching the market.

To appreciate how businesses operate, we must first get an idea of how prices are set in competitive markets. The next section, "Perfect Competition and Supply and Demand," begins by describing how markets establish prices in an environment of perfect competition.

Perfect competition and supply and demand

Under a mixed economy, such as we have in the United States, businesses make decisions about which goods to produce or services to offer and how they are priced. Because there are many businesses making goods or providing services, customers can choose among a wide array of products. The competition for sales among businesses is a vital part of our economic system. Economists have identified four types of competition—perfect competition, monopolistic competition, oligopoly, and monopoly. We'll introduce the first of these—perfect competition—in this section and cover the remaining three in the following section.

Perfect competition

Perfect competition exists when there are many consumers buying a standardized product from numerous small businesses. Because no seller is big enough or influential enough to affect price, sellers and buyers accept the going price. For example, when a commercial fisher brings his fish to the local market, he has little control over the price he gets and must accept the going market price.

The basics of supply and demand

To appreciate how perfect competition works, we need to understand how buyers and sellers interact in a market to set prices. In a market characterized by perfect competition, price is determined through the mechanisms of supply and demand. Prices are influenced both by the supply of products from sellers and by the demand for products by buyers.

To illustrate this concept, let's create a supply and demand schedule for one particular good sold at one point in time. Then we'll define demand and create a demand curve and define supply and create a supply curve. Finally, we'll see how supply and demand interact to create an equilibrium price—the price at which buyers are willing to purchase the amount that sellers are willing to sell.

Demand and the demand curve

Demand is the quantity of a product that buyers are willing to purchase at various prices. The quantity of a product that people are willing to buy depends on its price. You're typically willing to buy less of a product when prices rise and more of a product when prices fall. Generally speaking, we find products more attractive at lower prices, and we buy more at lower prices because our income goes further.

Using this logic, we can construct a demand curve that shows the quantity of a product that will be demanded at different prices. Let's assume that the diagram in Figure 7 represents the daily price and quantity of apples sold by farmers at a local market. Note that as the price of apples goes down, buyers' demand goes up. Thus, if a pound of apples sells for EUR 0.80, buyers will be willing to purchase only fifteen hundred pounds per day. But if apples cost only EUR 0.60 a pound, buyers will be willing to purchase two thousand pounds. At EUR 0.40 a pound, buyers will be willing to purchase twenty-five hundred pounds.

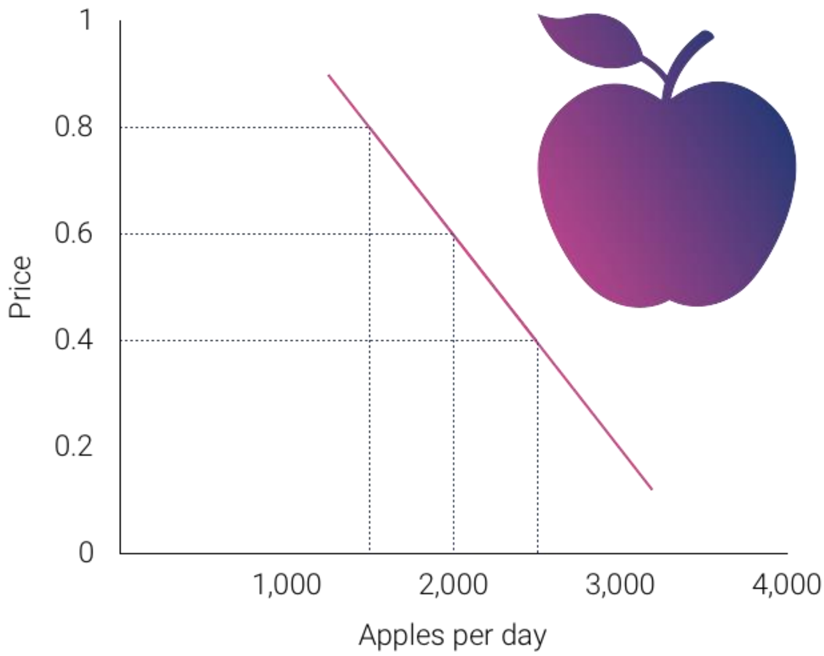

Figure 7: The Demand Curve

Supply and the supply curve

Supply is the quantity of a product that sellers are willing to sell at various prices. The quantity of a product that a business is willing to sell depends on its price. Businesses are more willing to sell a product when the price rises and less willing to sell it when prices fall. Again, this fact makes sense: businesses are set up to make profits, and there are larger profits to be made when prices are high.

Now we can construct a supply curve that shows the quantity of apples that farmers would be willing to sell at different prices, regardless of demand. As you can see in Figure 8, the supply curve goes in the opposite direction from the demand curve: as prices rise, the quantity of apples that farmers are willing to sell also goes up. The supply curve shows that farmers are willing to sell only a thousand pounds of apples when the price is EUR 0.40 a pound, two thousand pounds when the price is EUR 0.60, and three thousand pounds when the price is EUR 0.80.

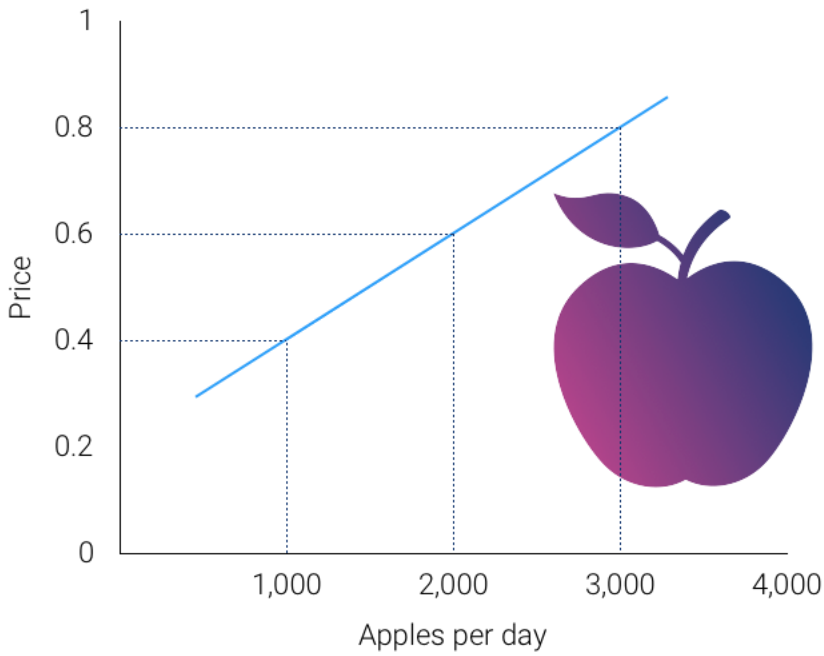

Figure 8: The Supply Curve

Equilibrium price

We can now see how the market mechanism works under perfect competition. We do this by plotting both the supply curve and the demand curve on one graph, as we've done in Figure 9. The point at which the two curves intersect is the equilibrium price.

You can see in Figure 9 that the supply and demand curves intersect at the price of EUR 0.60 and quantity of two thousand pounds. Thus, EUR 0.60 is the equilibrium price: at this price, the quantity of apples demanded by buyers equals the quantity of apples that farmers are willing to supply. If a single farmer tries to charge more than EUR 0.60 for a pound of apples, he won't sell very many because other suppliers are making them available for less. As a result, his profits will go down. If, on the other hand, a farmer tries to charge less than the equilibrium price of EUR 0.60 a pound, he will sell more apples but his profit per pound will be less than at the equilibrium price. With profit being the motive, there is no incentive to drop the price.

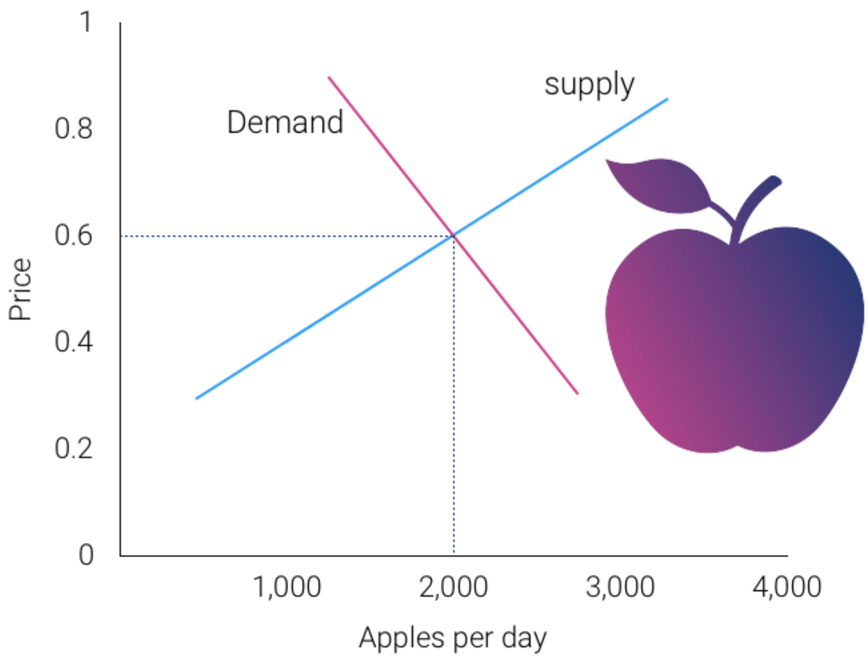

Figure 9: The Equilibrium Price

What have we learned in this discussion? Without outside influences, markets in an environment of perfect competition will arrive at an equilibrium point at which both buyers and sellers are satisfied. But we must be aware that this is a very simplistic example. Things are more complex in the real world. For one thing, markets don't always operate without outside influences. For example, if a government set an artificially low price ceiling on a product to keep consumers happy, we would not expect producers to produce enough to satisfy demand, resulting in a shortage. If government set prices high to assist an industry, sellers would likely supply more of a product than buyers need; in that case, there would be a surplus.

Circumstances also have a habit of changing. What would happen, for example, if incomes rose and buyers were willing to pay more for apples? The demand curve would change, resulting in an increase in equilibrium price. This outcome makes intuitive sense: as demand increases, prices will go up. What would happen if apple crops were larger than expected because of favorable weather conditions? Farmers might be willing to sell apples at lower prices rather than letting part of the crop spoil. If so, the supply curve would shift, resulting in another change in equilibrium price: the increase in supply would bring down prices.

Monopolistic competition, oligopoly, and monopoly

As mentioned previously, economists have identified four types of competition—perfect competition, monopolistic competition, oligopoly, and monopoly. Perfect competition was discussed in the last section; we'll cover the remaining three types of competition here.

Monopolistic competition

In monopolistic competition, we still have many sellers (as we had under perfect competition). Now, however, they don't sell identical products. Instead, they sell differentiated products—products that differ somewhat, or are perceived to differ, even though they serve a similar purpose. Products can be differentiated in a number of ways, including quality, style, convenience, location, and brand name. An example in this case might be toothpaste. Although many people are fiercely loyal to their favorites, most products in this category are quite similar and address the same consumer need. But what if there was a substantial price difference among products? In that case, many buyers would likely be persuaded to switch brands, at least on a trial basis.

How is product differentiation accomplished? Sometimes, it's simply geographical; you probably buy gasoline at the station closest to your home regardless of the brand. At other times, perceived differences between products are promoted by advertising designed to convince consumers that one product is different from an- other—and better than it. Regardless of customer loyalty to a product, however, if its price goes too high, the seller will lose business to a competitor. Under monopolistic competition, therefore, companies have only limited control over price.

Oligopoly

Oligopoly means few sellers. In an oligopolistic market, each seller supplies a large portion of all the products sold in the marketplace. In addition, because the cost of starting a business in an oligopolistic industry is usually high, the number of firms entering it is low. Companies in oligopolistic industries include such large-scale enterprises as automobile companies and airlines. As large firms supplying a sizable portion of a market, these companies have some control over the prices they charge. But there's a catch: because products are fairly similar, when one company lowers prices, others are often forced to follow suit to remain competitive. You see this practice all the time in the airline industry: When American Airlines announces a fare decrease, Continental, United Airlines, and others do likewise. When one automaker offers a special deal, its competitors usually come up with similar promotions.

Monopoly

In terms of the number of sellers and degree of competition, a monopoly lies at the opposite end of the spectrum from perfect competition. In perfect competition, there are many small companies, none of which can control prices; they simply accept the market price determined by supply and demand. In a monopoly, however, there's only one seller in the market. The market could be a geographical area, such as a city or a regional area, and doesn't necessarily have to be an entire country.

There are few monopolies in the United States because the government limits them. Most fall into one of two categories: natural and legal. Natural monopolies include public utilities, such as electricity and gas suppliers. Such enterprises require huge investments, and it would be inefficient to duplicate the products that they provide. They inhibit competition, but they're legal because they're important to society. In exchange for the right to conduct business without competition, they're regulated. For instance, they can't charge whatever prices they want, but they must adhere to government-controlled prices. As a rule, they're required to serve all customers, even if doing so isn't cost efficient.

A legal monopoly arises when a company receives a patent giving it exclusive use of an invented product or process. Patents are issued for a limited time, generally twenty years.[10] During this period, other companies can't use the invented product or process without permission from the patent holder. Patents allow companies a certain period to recover the heavy costs of researching and developing products and technologies. A classic example of a company that enjoyed a patent-based legal monopoly is Polaroid, which for years held exclusive ownership of instant-film technology.[11] Polaroid priced the product high enough to recoup, over time, the high

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[10] United States Patent and Trademark Office (2015). "General Information Concerning Patents." Retrieved from: http://www.uspto.gov/web/offices/pac/doc/general/index.html#laws

[11] Mary Bellis (2015). "Edwin Land and Polaroid Photography." About Money.com. Retrieved from: http://inventors.about.com/library/inventors/blpolaroid.htm

cost of bringing it to market. Without competition, in other words, it enjoyed a monopolistic position in regard to pricing.

Measuring the health of the economy

Every day, we are bombarded with economic news (at least if you watch the business news stations). We're told about things like unemployment, home prices, and consumer confidence trends. As a student learning about business, and later as a business manager, you need to understand the nature of the U.S. economy and the terminology that we use to describe it. You need to have some idea of where the economy is heading, and you need to know something about the government's role in influencing its direction.

Economic goals

The world's economies share three main goals:

- Growth
- High employment
- Price stability

Let's take a closer look at each of these goals, both to find out what they mean and to show how we determine whether they're being met.

Economic growth

One purpose of an economy is to provide people with goods and services—cars, computers, video games, houses, rock concerts, fast food, amusement parks. One way in which economists measure the performance of an economy is by looking at a widely used measure of total output called the gross domestic product (GDP). The GDP is defined as the market value of all goods and services produced by the economy in a given year. The GDP includes only those goods and services produced domestically; goods produced outside the country are excluded. The GDP also includes only those goods and services that are produced for the final user; intermediate products are excluded. For example, the silicon chip that goes into a computer (an intermediate product) would not count directly because it is included when the finished computer is counted. By itself, the GDP doesn't necessarily tell us much about the direction of the economy. But change in the GDP does. If the GDP (after adjusting for inflation, which will be discussed later) goes up, the economy is growing. If it goes down, the economy is contracting.

The business cycle

The economic ups and downs resulting from expansion and contraction constitute the business cycle. A typical cycle runs from three to five years but could last much longer. Though typically irregular, a cycle can be divided into four general phases of prosperity, recession, depression (which the cycle generally skips), and recovery:

- During prosperity, the economy expands, unemployment is low, in- comes rise, and consumers buy more products. Businesses respond by increasing production and offering new and better products.
- Eventually, however, things slow down. GDP decreases, unemployment rises, and because people have less money to spend, business revenues decline. This slowdown in economic activity is called a recession.
- Economists often say that we're entering a recession when GDP goes down for two consecutive quarters.
- Generally, a recession is followed by a recovery in which the economy starts growing again.

- If, however, a recession lasts a long time (perhaps a decade or so), while unemployment remains very high and production is severely curtailed, the economy could sink into a depression. Unlike for the term recession, economists have not agreed on a uniform standard for what constitutes a depression, though they are generally characterized by their duration. Though not impossible, it's unlikely that the United States will experience another severe depression like that of the 1930s. The federal government has a number of economic tools (some of which we'll discuss shortly) with which to fight any threat of a depression.

Full employment

To keep the economy going strong, people must spend money on goods and services. A reduction in personal expenditures for things like food, clothing, appliances, automobiles, housing, and medical care could severely reduce GDP and weaken the economy. Because most people earn their spending money by working, an important goal of all economies is making jobs available to everyone who wants one. In principle, full employment occurs when everyone who wants to work has a job. In practice, we say that we have full employment when about 95 percent of those wanting to work are employed.

The unemployment rate

Governmental institutions track unemployment and report the unemployment rate: the percentage of the labor force that's unemployed and actively seeking work. The unemployment rate is an important measure of economic health. It goes up during recessionary periods because companies are reluctant to hire workers when demand for goods and services is low. Conversely, it goes down when the economy is expanding and there is high demand for products and workers to supply them.

Figure 10 shows the unemployment rate in Austria between 1970 and 2018.

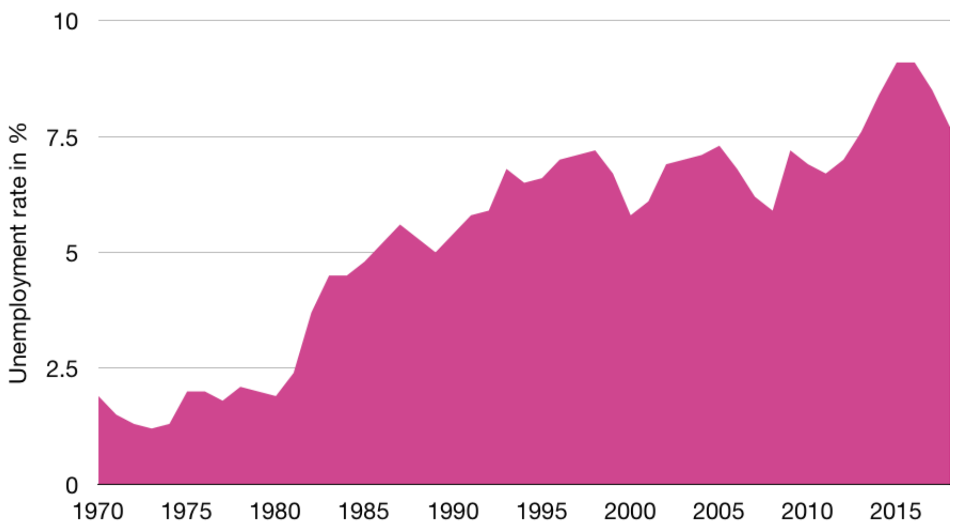

Figure 10: The Austrian unemployment rate, 1970-2018

Price stability

A third major goal of all economies is maintaining price stability. Price stability occurs when the average of the prices for goods and services either doesn't change or changes very little. Rapidly rising prices are troublesome for both individuals and businesses. For individuals, rising prices mean people have to pay more for the things they need. For businesses, rising prices mean higher costs, and, at least in the short run, businesses might have trouble passing on higher costs to consumers. When the overall price level goes up, we have inflation. Figure 11 shows the inflation trends in Austria since 1970. When the price level goes down (which rarely happens), we have deflation. A deflationary situation can also be damaging to an economy. When purchasers believe they can expect lower prices in the future, they may defer making purchases, which has the effect of slowing economic growth (GDP) accompanied by a rise in unemployment. Japan experienced a long period of deflation which contributed to economic stagnation in that country from which it is only now beginning to recover.

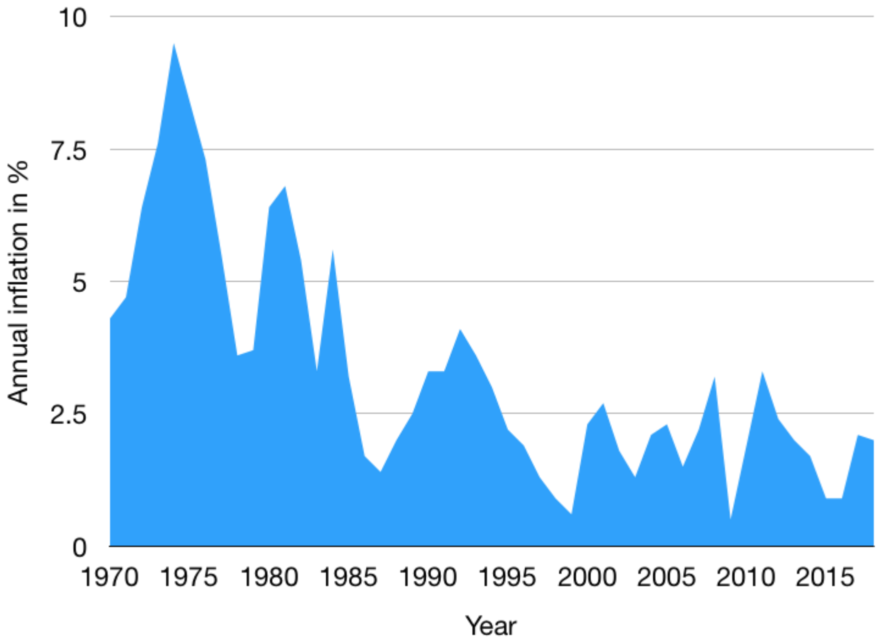

Figure 11: The U.S. Inflation Rate, 1960-2014

The consumer price index

The most widely publicized measure of inflation is the consumer price index (CPI). The CPI measures the rate of inflation by determining price changes of a hypothetical basket of goods, such as food, housing, clothing, medical care, appliances, automobiles, and so forth, bought by a typical household.

The CPI base value for Austria is regularly recalculated, most recently in 2015. Figure 12 shows the CPI values calculated on the basis of the CPI-66, i.e. on the basis of typical purchases in 1966. This means that products worth the equivalent of EUR 100 in 1966, cost EUR 115 in 1970. To buy the same typical goods in 2000, you had to spend EUR 375. The difference registers the effect of inflation. In fact, that's what an inflation rate is - the percentage change of a price index.

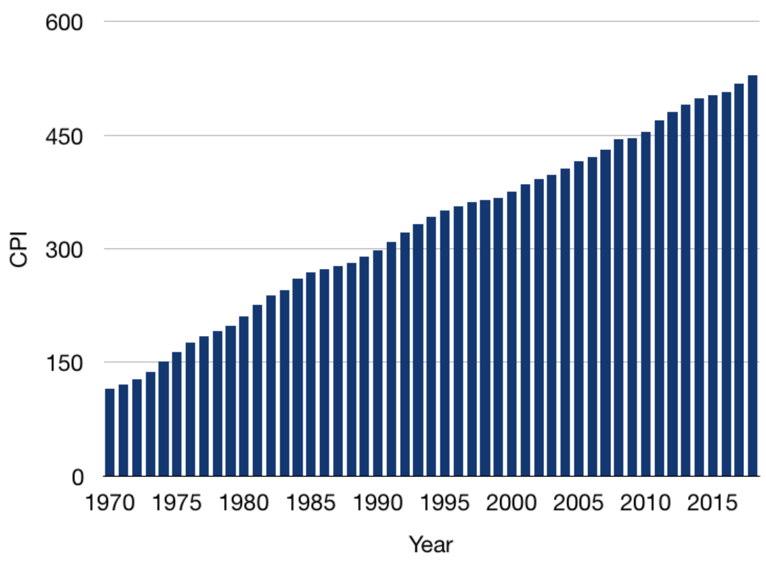

Figure 12: CPI Values, 1970-2018

Economic forecasting

In the previous section, we introduced several measures that economists use to assess the performance of the economy at a given time. By looking at changes in the GDP, for instance, we can see whether the economy is growing. The CPI allows us to gauge inflation. These measures help us understand where the economy stands today. But what if we want to get a sense of where it's headed in the future? To a certain extent, we can forecast future economic trends by analyzing several leading economic indicators.

Economic indicators

An economic indicator is a statistic that provides valuable information about the economy. There's no shortage of economic indicators, and trying to follow them all would be an overwhelming task. So in this chapter, we'll only discuss the general concept and a few of the key indicators.

Lagging and leading indicators

Economists use a variety of statistics to discuss the health of an economy. Statistics that report the status of the economy looking at past data are called lagging economic indicators. This type of indicator looks at trends to determine how strong an economy is and its direction. One such indicator is average length of unemployment. If unemployed workers have remained out of work for a long time, we may infer that the economy has been slow. Another lagging indicator is GDP growth. Even if the last several quarters have followed the same trend, though, there is no way to say with confidence that such a trend will necessarily continue.

Indicators that predict the status of the economy three to twelve months into the future are called leading economic indicators. If such an indicator rises, the economy is more likely to expand in the coming year. If it falls, the economy is more likely to contract. An example of a leading indicator is the number of permits obtained to build homes in a particular time period. If people intend to build more homes, they will be buying materials like lumber and appliances, and also employ construction workers. This type of indicator has a direct predictive value since it tells us something about what level of activity is likely in a future period.

In addition to housing, it is also helpful to look at indicators from sectors like labor and manufacturing. One useful indicator of the outlook for future jobs is the number of new claims for unemployment insurance. This measure tells us how many people recently lost their jobs. If it's rising, it signals trouble ahead because unemployed consumers can't buy as many goods and services as they could if they had paychecks. To gauge the level of goods to be produced in the future (which will translate into future sales), economists look at a statistic called average weekly manufacturing hours. This measure tells us the average number of hours worked per week by production workers in manufacturing industries. If it's on the rise, the economy will probably improve.

Since employment is such a key goal in any economy, the Bureau of Labor Statistics tracks total non-farm payroll employment from which the number of net new jobs created can be determined.

The Conference Board also publishes a consumer confidence index based on results of a monthly survey of five thousand U.S. households. The survey gathers consumers' opinions on the health of the economy and their plans for future purchases. It's often a good indicator of consumers' future buying intent.

Government's role in managing the economy

Monetary policy

Monetary policy is exercised by the Federal Reserve System ("the Fed"), which is empowered to take various actions that decrease or increase the money supply and raise or lower short-term interest rates, making it harder or easier to borrow money. When the Fed believes that inflation is a problem, it will use contractionary policy to decrease the money supply and raise interest rates. When rates are higher, borrowers have to pay more for the money they borrow, and banks are more selective in making loans. Because money is "tighter"—more expensive to borrow—demand for goods and services will go down, and so will prices. In any case, that's the theory.

The Fed will typically tighten or decrease the money supply during inflationary periods, making it harder to borrow money.

To counter a recession, the Fed uses expansionary policy to increase the money supply and reduce interest rates. With lower interest rates, it's cheaper to borrow money, and banks are more willing to lend it. We then say that money is "easy." Attractive interest rates encourage businesses to borrow money to expand production and encourage consumers to buy more goods and services. In theory, both sets of actions will help the economy escape or come out of a recession.

Fiscal policy

Fiscal policy relies on the government's powers of spending and taxation. Both taxation and government spending can be used to reduce or increase the total supply of money in the economy—the total amount, in other words, that businesses and consumers have to spend. When the country is in a recession, government policy is typically to increase spending, reduce taxes, or both. Such expansionary actions will put more money in the hands of businesses and consumers, encouraging businesses to expand and consumers to buy more goods and services. When the economy is experiencing inflation, the opposite policy is adopted: the government will decrease spending or increase taxes, or both. Because such contractionary measures reduce spending by businesses and consumers, prices come down and inflation eases.

Key takeaways

1. Economics is the study of the production, distribution, and consumption of goods and services.
2. Economists address these three questions: (1) What goods and services should be produced to meet consumer needs? (2) How should they be produced, and who should produce them? (3) Who should receive goods and services?
3. The answers to these questions depend on a country's economic system. The primary economic systems that exist today are planned and free-market systems.
4. In a planned system, such as communism and socialism, the government exerts control over the production and distribution of all or some goods and services.
5. In a free-market system, also known as capitalism, business is conducted with limited government involvement. Competition determines what goods and services are produced, how they are produced, and for whom.
6. When the market is characterized by perfect competition, many small companies sell identical products. The price is determined by supply and demand. Commodities like corn are an excellent example.
7. Supply is the quantity of a product that sellers are willing to sell at various prices. Producers will supply more of a product when prices are high and less when they're low.
8. Demand is the quantity of a product that buyers are willing to purchase at various prices; they'll buy more when the price is low and less when it's high.
9. In a competitive market, the decisions of buyers and sellers interact until the market reaches an equilibrium price—the price at which buyers are willing to buy the same amount that sellers are willing to sell.
10. There are three other types of competition in a free market system: monopolistic competition, oligopoly, and monopoly.
11. In monopolistic competition, there are still many sellers, but products are differentiated, e., differ slightly but serve similar purposes. By making consumers aware these differences, sellers exert some control over price.
12. In an oligopoly, a few sellers supply a sizable portion of products in the market. They exert some control over price, but because their products are similar, when one company lowers prices, the others follow.
13. In a monopoly, there is only one seller in the market. The "market" could be a specific geographical area, such as a city. The single seller is able to control prices.
14. All economies share three goals: growth, high employment, and price stability.
15. To get a sense of where the economy is headed in the future, we use statistics called economic indicators. Indicators that report the status of the economy a few months in the past are lagging Those that predict the status of the economy three to twelve months in the future are called leading indicators.

Chapter 3 – Management and leadership

Stephen Skripak, Anastasia Cortes, and Anita Walz

Learning objectives

1. Identify the four interrelated functions of management: planning, organizing, leading, and controlling.
2. Understand the process by which a company develops and implements a strategic plan.
3. Explain how managers direct others and motivate them to achieve company goals.
4. Describe the process by which a manager monitors operations and assesses performance.
5. Explain what benchmarking is and its importance for managing organizations.
6. Describe the skills needed to be a successful manager.

Noteworthy management

Consider this scenario: you're halfway through the semester and ready for midterms. You open your class notes and declare them "pathetic." You regret scribbling everything so carelessly and skipping class so many times. That's when it hits you: what if there was a note-taking service on campus? When you were ready to study for a big test, you could buy complete and legible class notes. You've heard that there are class-notes services at some larger schools, but there's no such thing on your campus. So you ask yourself, why don't I start a note-taking business? Your upcoming set of exams may not be salvageable, but after that, you'd always have great notes. And in the process, you could learn how to manage a business (isn't that what majoring in business is all about?).

You might begin by hiring a bunch of students to take class notes. Then the note takers will e-mail them to your assistant, who'll get them copied (on a special type of paper that can't be duplicated). The last step will be assembling packages of notes and, of course, selling them. You decide to name your company "Notes-4-You."

Figure 13: Management requires you to be both efficient and effective

It sounds like a great idea, but you're troubled by one question: why does this business need you? Do the note takers need a boss? Couldn't they just sell the notes themselves? This process could work, but it would work better if there was someone to oversee the operations: a manager—to make sure that the operations involved in preparing and selling notes were performed in both an effective and an efficient manner. You'd make the process **effective** by ensuring that the right things got done and that they all contributed to the success of the enterprise. You'd make the process **efficient** by ensuring that activities were performed in the right way and used the fewest possible resources.

What do managers do?

The management process

The effective performance of your business will require solid **management**: the process of planning, organizing, leading, and controlling resources to achieve specific goals. A **plan** enables you to take your business concept beyond the idea stage. It does not, however, get the work done. For that to happen, you have to **organize** things effectively. You'll have to put people and other resources in place to make things happen. And because your note-taking venture is supposed to be better off with you in charge, you need to be a **leader** who can motivate your people to do well. Finally, to know whether things are in fact going well, you'll have to **control** your operations—that is, measure the results and compare them with the results that you laid out in your plan. Figure 8.2 summarizes the interrelationship between planning and the other functions that managers perform. This chapter will explore planning, leading, and controlling in some detail.

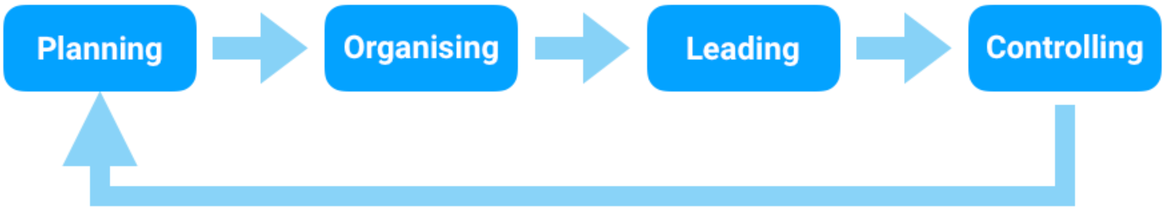

Figure 14: The Management Process

Planning

Without a plan, it's hard to succeed at anything. The reason is simple: if you don't know where you're going, you can't move forward. Successful managers decide where they want to be and then figure out how to get there; they set goals and determine the best way to achieve them. As a result of the planning process, everyone in the organization knows what should be done, who should do it, and how to do it.

Developing a strategic plan

Coming up with an idea—say, starting a note-taking business—is a good start, but it's only a start. Planning for it is a step forward. Planning begins at the highest level and works its way down through the organization. Step one is usually called strategic planning: the process of establishing an overall course of action. To begin this process, you should ask yourself a couple of very basic questions: why, for example, does the organization exist? What value does it create? Sam Walton posed these questions in the process of founding Wal-Mart: his new chain of stores would exist to offer customers the lowest prices with the best possible service.[12]

Once you've identified the purpose of your company, you're ready to take the remaining steps in the strategic-planning process:

- Write a mission statement that tells customers, employees, and others why your organization exists.
- Identify core values or beliefs that will guide the behavior of members of the organization.
- Assess the company's strengths, weaknesses, opportunities, and threats.
- Establish goals and objectives, or performance targets, to direct all the activities that you'll perform to achieve your mission.
- Develop and implement tactical and operational plans to achieve goals and objectives.

In the next few sections, we'll examine these components of the strategic-planning process.

Mission statement

As we saw in an earlier chapter, the **mission statement** describes the purpose of your organization—the reason for its existence. It tells the reader what the organization is committed to doing. It can be very concise, like the one from Mary Kay Inc. (the cosmetics company): "To enrich the lives of women around the world."[13] Or it can be as detailed as the one from Harley-Davidson: "We fulfill dreams inspired by the many roads of the world by providing extraordinary

[12] Wal Mart (2016). "Our Story." Walmart.com. Retrieved from: http://corporate.walmart.com/our-story/our-history
[13] Mary Kay (2016). "Corporate Careers: Discover what you love about Mary Kay." MaryKay.com. Retrieved from: http://www.marykay.com/en-US/About-Mary-Kay/EmploymentMaryKay

motorcycles and customer experiences. We fuel the passion for freedom in our customers to express their own individuality."[14]

A mission statement for Notes-4-You could be the following: "To provide high-quality class notes to college students." On the other hand, you could prepare a more detailed statement that explains what the company is committed to doing, who its customers are, what its focus is, what goods or services it provides, and how it serves its customers.

It is worth noting that some companies no longer use mission statements, preferring to communicate their reason for being in other manners.

Core values

Whether or not your company has defined a mission, it is important to identify what your organization stands for in terms of its values and the principles that will guide its actions. In Chapter 3 on Business Ethics and Social Responsibility, we explained that the small set of guiding principles that you identify as crucial to your company are known as **core values**—fundamental beliefs about what's important and what is and isn't appropriate in conducting company activities. Core values affect the overall planning processes and operations. At Volvo, three values— safety, quality, and environmental care—define the firm's "approach to product development, design and production."[15] Core values should also guide the behavior of every individual in the organization. At Coca-Cola, for instance, the values of leadership, collaboration, integrity, accountability, passion, diversity and quality tell employees exactly what behaviors are acceptable.[16] Companies communicate core values to employees and hold them accountable for putting them into practice by linking their values to performance evaluations and compensation.

In choosing core values for Notes-4-You, you're determined to be unique. After some thought, you settle on teamwork, trust, and dependability. Why these three? As you plan your business, you realize that it will need a workforce that functions as a team, trusts each other, and can be depended on to satisfy customers. In building your workforce, you'll seek employees who'll embrace these values.

Conduct a SWOT analysis

The next step in the strategic-planning process is to assess your company's fit with its environment. A common approach to environmental analysis is matching the strengths of your business with the opportunities available to it. It's called **SWOT analysis** because it calls for analyzing an organization's Strengths, Weaknesses, Opportunities, and Threats. It begins with an examination of **external factors** that could influence the company in either a positive or a negative way. These could include economic conditions, competition, emerging technologies, laws and regulations, and customers' expectations.

One purpose of assessing the external environment is to identify both **opportunities** that could benefit the company and **threats** to its success. For example, a company that manufactures children's bicycle helmets would view a change in federal law requiring all children to wear helmets as an opportunity. The news that two large sports-equipment companies were coming out with bicycle helmets would be a threat.

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[14] Harley Davidson (2016). "About Harley Davidson." Harleydavidson.com. Retrieved from: http://www.harley-davidson.com/content/h-d/en_US/company.html

[15] Volvo Group (2016). "Volvo Group Global: Our Values." Volvogroup.com. Retrieved from: http://www.volvogroup.com/group/global/en-gb/volvo%20group/ourvalues/Pages/volvovalues.aspx

[16] Coca Cola Company (2016). "Our Company: Vision, Mission, and Values." Cocacola.com. Retrieved from: http://www.coca-colacompany.com/our-company/mission-vision-values

The next step is to evaluate the company's strengths and weaknesses, internal factors that could influence company performance in either a positive or negative way. Strengths might include a motivated workforce, state-of-the-art technology, impressive managerial talent, or a desirable location. The opposite of any of these strengths could signal a potential weakness (poor workforce, obsolete technology, incompetent management, or poor location). Armed with a good idea of internal strengths and weaknesses, as well as external opportunities and threats, managers will be better positioned to capitalize on opportunities and strengths. Likewise, they want to improve on any weak areas and protect the organization from external threats.

For example, Notes-4-You might say that by providing excellent service at a reasonable price while we're still small, it can solidify its position on campus. When the market grows due to increases in student enrollment, the company will have built a strong reputation and be in a position to grow. So even if a competitor comes to campus (a threat), the company expects to be the preferred supplier of class notes. This strategy will work only if the note-takers are dependable and if the process does not alienate the faculty or administration.

Set goals and objectives

Your mission statement affirms what your organization is generally committed to doing, but it doesn't tell you how to do it. So the next step in the strategic-planning process is establishing goals and objectives. Goals are major accomplishments that the company wants to achieve over a long period. Objectives are shorter-term performance targets that direct the activities of the organization toward the attainment of a goal. They should be clearly stated, achievable, and measurable: they should give target dates for the completion of tasks and stipulate who's responsible for taking necessary actions.[17]

An organization will have a number of goals and related objectives. Some will focus on financial measures, such as profit maximization and sales growth. Others will target operational efficiency or quality control. Still others will govern the company's relationships with its employees, its community, its environment, or all three.

Finally, goals and objectives change over time. As a firm reassesses its place in its business environment, it rethinks not only its mission but also its approach to fulfilling it. The reality of change was a major theme when the late McDonald's CEO Jim Cantalupo explained his goal to revitalize the company:

> "The world has changed. Our customers have changed. We have to change too. Growth comes from being better, not just expanding to have more restaurants. The new McDonald's is focused on building sales at existing restaurants rather than on adding new restaurants. We are introducing a new level of discipline and efficiency to all aspects of the business and are setting a new bar for performance."[18]

This change in focus was accompanied by specific performance objectives—annual sales growth of 3 to 5 percent and income growth of 6 to 7 percent at existing restaurants, plus a five-point improvement (based on customer surveys) in speed of service, friendliness, and food quality.

In setting strategic goals and performance objectives for Notes-4-You, you should keep things simple. Because you need to make money to stay in business, you could include a financial goal

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[17] Scott Safranski and Ik-Whan Kwon (1991). "Strategic Planning for the Growing Business." U.S. Small Business Administration. Retrieved from: http://webharvest.gov/peth04/20041105092332/http://sba.gov/library/pubs/eb-6.pdf
[18] Franchise Bison (2003). ""McDonald's Announces Plans to Revitalize Its Worldwide Business and Sets New Financial Targets." Franchisebison.com. Retrieved from: http://www.bison1.com/press_mcdonalds_04072003

(and related objectives). Your mission statement promises “high-quality, dependable, competitively priced class notes,” so you could focus on the quality of the class notes that you'll be taking and distributing. Finally, because your mission is to serve students, one goal could be customer oriented. Your list of goals and objectives might look like this:

- Goal 1: Achieve a 10 percent return on sales in your first five years.
- *Objective*: Sales of EUR 20,000 and profit of EUR 2,000 for the first 12 months of operation.
- Goal 2: Produce a high-quality product.
- *Objective*: First-year satisfaction scores of 90 percent or higher on quality of notes (based on survey responses on understandability, readability, and completeness).
- Goal 3: Attain 98 percent customer satisfaction by the end of your fifth year.
- *Objective*: Making notes available within two days after class, 95 percent of the time.

Tactical plans

The overall plan is broken down into more manageable, shorter-term components called tactical plans. These plans specify the activities and allocation of resources (people, equipment, money) needed to implement the strategic plan over a given period. Often, a long-range strategic plan is divided into several tactical plans; a five-year strategic plan, for instance, might be implemented as five one-year tactical plans.

Operational plans

The tactical plan is then broken down into various operational components that provide detailed action steps to be taken by individuals or groups to implement the tactical and strategic plans. Operational plans cover only a brief period—say, a month or two. At Notes-4-You, note-takers might be instructed to submit typed class notes five hours earlier than normal on the last day of the semester (an operational guideline). The goal is to improve the customer-satisfaction score on dependability (a tactical goal) and, as a result, to earn the loyalty of students through attention to customer service (a strategic goal).

Figure 15

Plan for Contingencies and Crises

Even with great planning, things do not always turn out the way they are supposed to. Perhaps your plans were flawed, or maybe something in the environment shifted unexpectedly.

Successful managers anticipate and plan for the unexpected. Dealing with uncertainty requires contingency planning and crisis management.

Contingency planning

With contingency planning, managers identify those aspects of the business that are most likely to be adversely affected by change. Then, they develop alternative courses of action in case an anticipated change does occur. You engage in contingency planning any time you develop a backup or fallback plan.

Crisis management

Organizations also face the risk of encountering crises that require immediate attention. Rather than waiting until such a crisis occurs and then scrambling to figure out what to do, many firms practice crisis management. Some, for instance, set up teams trained to deal with emergencies. Members gather information quickly and respond to the crisis while everyone else carries out his or her normal duties. The team also keeps the public, the employees, the press, and government officials informed about the situation and the company's response to it.[19]

An example of how to handle crisis management involves Wendy's. After learning that a woman claimed she found a fingertip in a bowl of chili she bought at a Wendy's restaurant in San Jose, California, the company's public relations team responded quickly. Within a few days, the company announced that the finger didn't come from an employee or a supplier. Soon after, the police arrested the woman and charged her with attempted grand larceny for lying about how the finger got in her bowl of chili and trying to extort $2.5 million from the company. But the crisis wasn't over for Wendy's. The incident was plastered all over the news as a grossed-out public sought an answer to the question, "Whose finger is (or was) it?" A $100,000 reward was offered by Wendy's to anyone with information that would help the police answer this question. The challenge Wendy's faced was how to entice customers to return to its fifty San Francisco–area restaurants (where sales had plummeted) while keeping a low profile nationally. Wendy's accomplished this objective by giving out free milkshakes and discount coupons to customers in the affected regions and, to avoid calling attention to the missing finger, by making no changes in its national advertising. The crisis-management strategy worked and the story died down (though it flared up temporarily when the police arrested the woman's husband, who allegedly bought the finger from a coworker who had severed it in an accident months earlier).[20]

Figure 16: A Wendy's Restaurant

[19] Brian Perkins (2000). "Defining Crisis Management." Wharton Magazine. Retrieved from: http://whartonmagazine.com/issues/summer-2000/reunion-2000/
[20] Matt Richtel (2005). "Wendy's Gets a Break, But Still Has Work Ahead of it." The New York Times. Retrieved from: http://www.nytimes.com/2005/04/29/business/media/wendys-gets-a-break-but-still-has-work-ahead-of-it.html?_r=0

Even with crisis-management plans in place, however, it's unlikely that most companies will emerge from a potentially damaging episode as unscathed as Wendy's did. For one thing, the culprits in the Wendy's case were caught, and the public was willing to forgive an organization it viewed as a victim. Given the current public distrust of corporate behavior, however, companies whose reputations have suffered due to questionable corporate judgment usually don't fare as well. These companies include the international oil company, BP, whose CEO, Tony Hayward, did a disastrous job handling the Gulf of Mexico crisis. A BP-controlled oil rig exploded in the Gulf of Mexico, killing eleven workers and creating the largest oil spill in U.S. history. Hayward's lack of sensitivity will be remembered forever; particularly his response to a reporter's question on what he would tell those whose livelihoods were ruined: "We're sorry for the massive disruption it's caused their lives. There's no one who wants this over more than I do. I would like my life back." His comment was obviously upsetting to the families of the eleven men who lost their lives on the rig.[21] Then, there are the companies at which executives have crossed the line between the unethical to the downright illegal—Arthur Andersen, Enron, and Bernard L. Madoff Investment Securities, to name just a few. Given the high risk associated with a crisis, it should come as no surprise that contemporary managers spend more time anticipating crises and practicing their crisis-management responses.

Figure 17: BP's Deepwater Horizon oil rig on fire in the Gulf of Mexico in 2010

Leading

The third management function is leading—providing focus and direction to others and motivating them to achieve organizational goals. As owner and president of Notes-4-You, you might think of yourself as an orchestra leader. You have given your musicians (employees) their sheet music (plans). You've placed them in sections (departments) and arranged the sections (organizational structure) so the music will sound as good as possible. Now your job is to tap your baton and lead the orchestra so that its members make beautiful music together.[22]

Leadership styles

It's fairly easy to pick up a baton, cue each section, and strike up the band; but it doesn't mean the music will sound good. What if your cues are ignored or misinterpreted or ambiguous? Maybe your musicians don't like your approach to making music and will just walk away. On top of everything else, you don't simply want to make music: you want to inspire your musicians to

[21] Jacqui Goddard (2010). "Embattled BP Chief: I Want my Life Back." The Times (London). Retrieved from: http://www.thetimes.co.uk/tto/news/article2534734.ece
[22] John Reh (n.d.). "Management 101." About Money. Retrieved from: http://management.about.com/cs/generalmanagement/a/Management101.htm

make great music. How do you accomplish this goal? How do you become an effective leader, and what style should you use to motivate others to achieve organizational goals?

Unfortunately, there are no definitive answers to questions like these. Over time, every manager refines his or her own leadership style, or way of interacting with and influencing others. Despite a vast range of personal differences, leadership styles tend to reflect one of the following approaches to leading and motivating people: the autocratic, the democratic (also known as participative), or the free rein.

- Autocratic style. Managers who have developed an autocratic leadership style tend to make decisions without soliciting input from subordinates. They exercise authority and expect subordinates to take responsibility for performing the required tasks without undue explanation.
- Democratic style. Managers who favor a democratic leadership style generally seek input from subordinates while retaining the authority to make the final decisions. They're also more likely to keep subordinates informed about things that affect their work.
- Free-rein style. In practicing a free rein leadership style, managers adopt a "hands-off" approach and provide relatively little direction to subordinates. They may advise employees but usually give them considerable freedom to solve problems and make decisions on their own.

At first glance, you'd probably not want to work for an autocratic leader. After all, most people don't like to be told what to do without having any input. Many like the idea of working for a democratic leader; it's flattering to be asked for your input. And though working in a free rein environment might seem a little unsettling at first, the opportunity to make your own decisions is appealing to many people. Each leadership style can be appropriate in certain situations.

To illustrate, let's say that you're leading a group of fellow students in a team project for your class. Are there times when it would be best for you to use an autocratic leadership style? What if your team was newly formed, unfamiliar with what needs to be done, under a tight deadline, and looking to you for direction? In this situation, you might find it appropriate to follow an autocratic leadership style (on a temporary basis) and assign tasks to each member of the group. In an emergency situation, such as a fire, or in the final seconds of a close ball game, there is generally not time for debate – the leader or coach must make a split second decision that demands an autocratic style.

But since most situations are non-emergency and most people prefer the chance to give input, the democratic leadership style is often favored. People are simply more motivated and feel more ownership of decisions (i.e., buy-in) when they have had a chance to offer input. Note that when using this style, the leader will still make the decision in most cases. As long as their input is heard, most people accept that it is the leader's role to decide in cases where not everyone agrees.

How about free rein leadership? Many people function most effectively when they can set their own schedules and do their work in the manner they prefer. It takes a great deal of trust for a manager to employ this style. Some managers start with an assumption of trust that is up to the employee to maintain through strong performance. In other cases, this trust must be earned over a period of time. Would this approach always work with your study group? Obviously not. It will work if your team members are willing and able to work independently and welcome the chance to make decisions. On the other hand, if people are not ready to work responsibly to their best of their abilities, using the free rein style could cause the team to miss deadlines or do poorly on the project.

The point being made here is that no one leadership style is effective all the time for all people or in all corporate cultures. While the democratic style is often viewed as the most appropriate (with the free rein style a close second), there are times when following an autocratic style is essential. Good leaders learn how to adjust their styles to fit both the situation and the individuals being directed.

Transformational leadership

Theories on what constitutes effective leadership evolve over time. One theory that has received a lot of attention in the last decade contrasts two leadership styles: transactional and transformational. So-called **transactional leaders** exercise authority based on their rank in the organization. They let subordinates know what's expected of them and what they will receive if they meet stated objectives. They focus their attention on identifying mistakes and disciplining employees for poor performance. By contrast, **transformational leaders** mentor and develop subordinates, providing them with challenging opportunities, working one-on-one to help them meet their professional and personal needs, and encouraging people to approach problems from new perspectives. They stimulate employees to look beyond personal interests to those of the group.

So, which leadership style is more effective? You probably won't be surprised by the opinion of most experts. In today's organizations, in which team building and information sharing are important and projects are often collaborative in nature, transformational leadership has proven to be more effective. Modern organizations look for managers who can develop positive relationships with subordinates and motivate employees to focus on the interests of the organization. Leaders who can be both transactional and transformational are rare, and those few who have both capacities are very much in demand.[23]

Controlling

Let's pause for a minute and reflect on the management functions that we've discussed so far—planning, organizing, and leading. As founder of Notes-4-You, you began by establishing plans for your new company. You defined its mission and set objectives, or performance targets, which you needed to meet in order to achieve your mission. Then, you organized your company by allocating the people and resources required to carry out your plans. Finally, you provided focus and direction to your employees and motivated them to achieve organizational objectives. Is your job finished? Can you take a well-earned vacation? Unfortunately, the answer is no: your work has just begun. Now that things are rolling along, you need to monitor your operations to see whether everything is going according to plan. If it's not, you'll need to take corrective action. This process of comparing actual to planned performance and taking necessary corrective action is called controlling.

A five-step control process

You can think of the **control function** as the five-step process outlined in Figure 8.7. Let's see how this process might work at Notes-4-You. Let's assume that, after evaluating class enrollments, you estimate that you can sell one hundred notes packages per month to

[23] Sarah Burke and Karen M. Collins, (2001). "Gender differences in leadership styles and management skills." Women in Management Review. PP.244 – 257

students taking a popular sophomore-level geology course. So you set your standard at a hundred units. At the end of the month, however, you look over your records and find that you sold only eighty. In talking with your salespeople, you learn why you came up twenty packages short: it turns out that the copy machine broke down so often that packages frequently weren't ready on time. You immediately take corrective action by increasing maintenance on the copy machine.

Figure 18

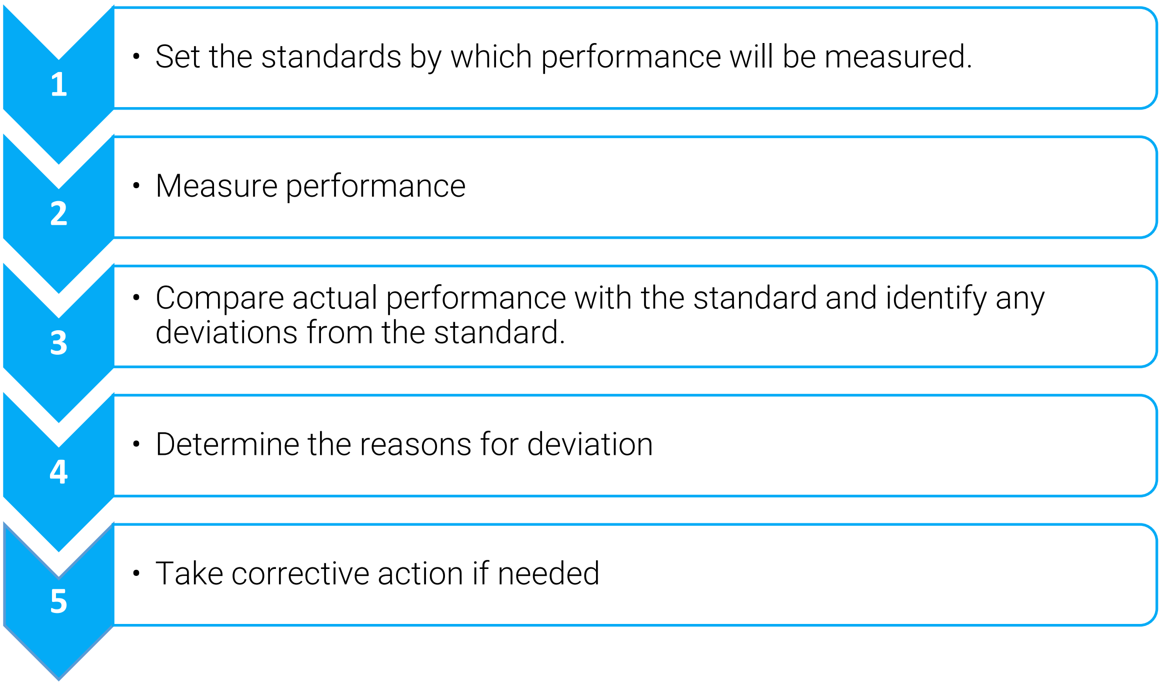

Figure 8.7: The Control Process

Now, let's try a slightly different scenario. Let's say that you still have the same standard (one hundred packages) and that actual sales are still eighty packages. In investigating the reason for the shortfall, you find that you overestimated the number of students taking the geology course. Calculating a more accurate number of students, you see that your original standard—estimated sales—was too high by twenty packages. In this case, you should adjust your standards to reflect expected sales of eighty packages.

In both situations, your control process has been helpful. In the first instance, you were alerted to a problem that cut into your sales. Correcting this problem would undoubtedly increase sales and, therefore, profits. In the second case, you encountered a defect in your planning and learned a good managerial lesson: plan more carefully.

Benchmarking

Benchmarking could be considered as a specialized kind of control activity. Rather than controlling a particular aspect of performance (say, defects for a specific product), benchmarking aims to improve a firm's overall performance. The process of benchmarking involves comparisons to other organizations' practices and processes with the objective of learning and improvement in both efficiency and effectiveness. Benchmarking exercises can be conducted in a number of ways:

- Organizations often monitor publicly available information to keep tabs on the competition. Annual reports, news articles, and other sources are monitored closely in order to stay aware of the latest developments. In academia, universities often use published rankings tables to see how their programs compare on the basis of standardized test scores, salaries of graduates, and other important dimensions.
- Organizations may also work directly with companies in unrelated industries in order to compare those functions of the business which are similar. A manufacture of aircraft would not likely have a great deal in common with a company making engineered plastics, yet both have common functions such as accounting, finance, information technology, and human resources. Companies can exchange ideas that help each other improve efficiency, and often at a very low cost to either.
- In order to compare more directly to competition without relying solely on publicly available data, companies may enter into benchmarking consortiums in which an outside consultant would collect key data from all participants, anonymize it, and then share the results with all participants. Companies can then gauge how they compare to others in the industry without revealing their own performance to others.

Managerial skills

To be a successful manager, you'll have to master a number of skills. To get an entry- level position, you'll have to be technically competent at the tasks you're asked to perform. To advance, you'll need to develop strong interpersonal and conceptual skills. The relative importance of different skills varies from job to job and organization to organization, but to some extent, you'll need them all to forge a managerial career.

Throughout your career, you'll also be expected to communicate ideas clearly, use your time efficiently, and reach sound decisions.

Technical skills

You'll probably be hired for your first job based on your technical skills—the ones you need to perform specific tasks—and you'll use them extensively during your early career. If your college major is accounting, you'll use what you've learned to prepare financial statements. If you have a marketing degree and you join an ad agency, you'll use what you know about promotion to prepare ad campaigns. Technical skills will come in handy when you move up to a first-line managerial job and oversee the task performance of subordinates. Technical skills, though developed through job training and work experience, are generally acquired during the course of your formal education.

Interpersonal skills

As you move up the corporate ladder, you'll find that you can't do everything yourself: you'll have to rely on other people to help you achieve the goals for which you're responsible. That's why interpersonal skills, also known as relational skills—the ability to get along with and motivate other people—are critical for managers in mid-level positions. These managers play a pivotal role because they report to top-level managers while overseeing the activities of first-line managers. Thus, they need strong working relationships with individuals at all levels and in all areas. More than most other managers, they must use "people skills" to foster teamwork, build trust, manage conflict, and encourage improvement.[24]

Conceptual skills

Managers at the top, who are responsible for deciding what's good for the organization from the broadest perspective, rely on **conceptual skills**—the ability to reason abstractly and analyze complex situations. Senior executives are often called on to "think outside the box"—to arrive at creative solutions to complex, sometimes ambiguous problems. They need both strong analytical abilities and strong creative talents.

Communication skills

Effective **communication skills** are crucial to just about everyone. At all levels of an organization, you'll often be judged on your ability to communicate, both orally and in writing. Whether you're talking informally or making a formal presentation, you must express yourself clearly and concisely. Talking too loudly, rambling, and using poor grammar reduce your ability to influence others, as does poor written communication. Confusing and error-riddled documents (including e-mails) don't do your message any good, and they will reflect poorly on you.[25]

Time-Management skills

Managers face multiple demands on their time, and their days are usually filled with interruptions. Ironically, some technologies that were supposed to save time, such as voicemail and e-mail, have actually increased workloads. Unless you develop certain **time-management skills**, you risk reaching the end of the day feeling that you've worked a lot but accomplished little. What can managers do to ease the burden? Here are a few common-sense suggestions:

- Prioritize tasks, focusing on the most important things first.
- Set aside a certain time each day to return phone calls and answer e-mail.
- Delegate routine tasks.
- Don't procrastinate.
- Insist that meetings start and end on time, and stick to an agenda.
- Eliminate unnecessary paperwork.[26]

Decision-making skills

Every manager is expected to make decisions, whether alone or as part of a team. Drawing on your **decision-making skills** is often a process in which you must define a problem, analyze possible solutions, and select the best outcome. As luck would have it, because the same process is good for making personal decisions, we'll use a personal example to demonstrate the process approach to decision making. Consider the following scenario: you're upset because your midterm grades are much lower than you'd hoped. To make matters worse, not only are you in trouble academically, but also the other members of your business-project team are annoyed because you're not pulling your weight. Your lacrosse coach is very upset because you've missed too many practices, and members of the mountain-biking club of which you're

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[24] Brian Perkins (2000). "Defining Crisis Management." Wharton Magazine. Retrieved from: http://whartonmagazine.com/issues/summer-2000/reunion-2000/

[25] Brian L. Davis et al. (1992). Successful Manager's Handbook: Development Suggestions for Today's Managers. Minneapolis: Personnel Decisions Inc. P. 189.

[26] Ibid.

supposed to be president are talking about impeaching you if you don't show up at the next meeting. And your significant other is feeling ignored.

A six-step approach to decision making

Assuming that your top priority is salvaging your GPA, let's tackle your problem by using a six-step approach to solving problems that don't have simple solutions. We've summarized this model in Figure 8.8[27]

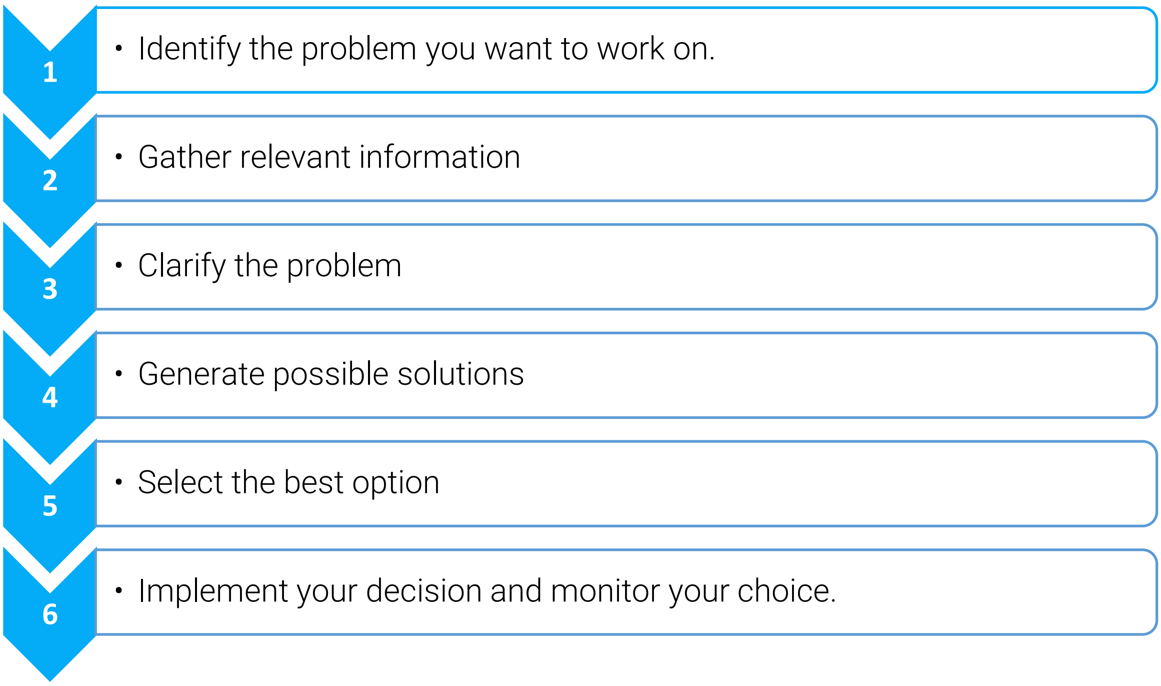

Figure 19: The problem solving and decision making process

Identify the problem you want to work on

Step one is getting to know your problem, which you can formulate by asking yourself a basic question: how can I improve my grades?

Gather relevant data

Step two is gathering information that will shed light on the problem. Let's rehash some of the relevant information that you've already identified: (a)you did poorly on your finals because you didn't spend enough time studying; (b) you didn't study because you went to see your girlfriend (who lives about three hours from campus) over the weekend before your exams (and on most other weekends, as a matter of fact); (c) what little studying you got in came at the expense of your team project and lacrosse practice; and (d) while you were away for the weekend, you forgot to tell members of the mountain-biking club that you had to cancel the planned meeting.

Clarify the problem

Once you review all the given facts, you should see that your problem is bigger than simply getting your grades up; your life is pretty much out of control. You can't handle everything to which you've committed yourself. Something has to give. You clarify the problem by summing it up with another basic question: what can I do to get my life back in order?

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[27] Shari Caudron (1998). "Six Steps in Creative Problem Solving." Controller Magazine. P. 38. Caudron describes a systematic approach developed by Roger L. Firestien, president of Innovation Systems Group, Williamsville, NY.

Generate possible solutions

Let's say that you've come up with the following possible solutions to your problem: (a) quit the lacrosse team, (b) step down as president of the mountain-biking club, (c) let team members do your share of work on the business project, and (d) stop visiting your significant other so frequently. The solution to your main problem—how to get your life back in order—will probably require multiple actions.

Select the best option

This is clearly the toughest part of the process. Working your way through your various options, you arrive at the following conclusions: (a) you can't quit the lacrosse team because you'd lose your scholarship; (b) you can resign your post in the mountain-biking club, but that won't free up much time; (c) you can't let your business-project team down (and besides, you'd just get a low grade); and (d) she wouldn't like the idea, but you could visit your girlfriend, say, once a month rather than once a week. So what's the most feasible (if not necessarily perfect) solution? Probably visiting your significant other once a month and giving up the presidency of the mountain-biking club.

Implement your decision and monitor your choice

When you call your girlfriend, you're pleasantly surprised to find that she understands. The vice president is happy to take over the mountain-biking club. After the first week, you're able to attend lacrosse practice, get caught up on your team business project, and catch up in all your other classes. The real test of your solution will be the results of the semester's finals.

Key takeaways

1. **Management** must include both **efficiency** (accomplishing goals using the fewest resources possible) and **effectiveness** (accomplishing goals as accurately as possible).
2. The management process has four **functions**: **planning**, **organizing**, **leading**, and **controlling**.
3. **Planning** for a business starts with **strategic planning**—the process of establishing an overall course of action.
4. Management first identifies its **purposes**, creates a **mission statement**, and defines its **core values**.
5. A **SWOT analysis** assesses the company's strengths and weaknesses and its fit with the external environment.
6. **Goals and objectives**, or performance targets, are established to direct company actions, and **tactical plans** and **operational plans** implement objectives.
7. A manager's **leadership style** varies depending on the manager, the situation, and the people being directed. There are several management styles.
    1. An **autocratic** manager tends to make decisions without input and expects subordinates to follow instructions.
    2. Managers who prefer a **democratic** style seek input into decisions.
    3. A **free rein** manager provides no more guidance than necessary and lets subordinates make decisions and solve problems.
    4. **Transactional** style managers exercise authority according to their rank in the organization, let subordinates know what's expected of them, and step in when mistakes are made.
    5. **Transformational** style managers mentor and develop subordinates and motivate them to achieve organizational goals.
8. The **control process** can be viewed as a five-step process: (1) establish standards, (2) **measure** performance, (3) **compare** actual performance with standards and identify any deviations, (4) **determine the reason** for deviations, and (5) **take corrective action** if needed.
9. **Benchmarking** is a process for improving overall company efficiency and effectiveness by comparing performance to competitors.
10. Top managers need strong **conceptual skills**, while those at midlevel need good **interpersonal skills** and those at lower levels need **technical skills**.
11. All managers need strong c**ommunication, decision-making,** and **time- management skills**.

Chapter 4 – Accounting and financial information

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Learning objectives

1. Define accounting and explain the differences between managerial accounting and financial accounting.
2. Identify some of the users of accounting information and explain how they use it.
3. Explain the function of the income statement.
4. Explain the function of the balance sheet.
5. Calculate a break-even point given the necessary information.
6. Evaluate a company's performance using financial statements and ratio analysis.

Figure 20: Apple Headquarters in Cupertino, California

Apple Inc. is the most valuable company in the world. This statement is based on market value, which in June 2016 was roughly $500 billion. Although markets can fluctuate, sometimes wildly, if you are reading this chapter, it is not unlikely that Apple will have retained its leadership position. Its value as of June 2016 was more than $40 billion greater than that of the next largest company, Alphabet, the parent company of Google. Apple has briefly ceded the leadership position to Alphabet on a couple of occasions, but for the most part, it has been the leader for quite some time.[28]

You may wonder what kind of information is used to make these determinations. How does the market know that Apple should be valued more than $100 billion higher than Exxon-Mobil, for example?[29] Do investors just make their decisions on instinct? Well, some do, but it's not a formula for sustained success. In most cases, in deciding how much to pay for a company, investors rely on published accounting and financial information released by publicly-traded companies. This chapter will introduce you to the subject of accounting and financial information so you can begin to get an understanding for how the valuation process works.

The role of accounting

Accounting is often called "the language of business" because it communicates so much of the information that owners, managers, and investors need to evaluate a company's financial performance. These people are stakeholders in the business—they're interested in its activities because they're affected by them. The financial futures of owners and other investors may depend heavily on strong financial performance from the business, and when performance is poor, managers may be replaced or laid off in a downsizing. In fact, a key purpose of accounting is to help stakeholders make better business decisions by providing them with financial information. You shouldn't try to run an organization or make investment decisions without accurate and timely financial information, and it is the accountant who prepares this information. More importantly, accountants make sure that stakeholders understand the meaning of financial information, and they work with both individuals and organizations to help them use financial information to deal with business problems. Actually, collecting all the numbers is the easy part. The hard part is analyzing, interpreting, and communicating the information. Of course, you also have to present everything clearly while effectively interacting with people from every business discipline. In any case, we're now ready to define accounting

[28] Financial data for the comparison from Yahoo Finance (2016). Apple data retrieved from: https://finance.yahoo.com/q?s=aapl&fr=uh3_finance_web&uhb=uhb2, and Alphabet data retrieved from: https://finance.yahoo.com/q?uhb=uhb2&fr=uh3_finance_vert_gs&type=2button&s=GOOG%2C Comparison date: June 27, 2016.
[29] Exxon Mobil data retrieved from: http://finance.yahoo.com/q?s=XOM. Comparison date: June 27, 2016.

as the process of measuring and summarizing business activities, interpreting financial information, and communicating the results to management and other decision makers.

Fields of accounting

Accountants typically work in one of two major fields. Management accountants provide information and analysis to decision makers inside the organization in order to help them run it. Financial accountants furnish information to individuals and groups both inside and outside the organization in order to help them assess its financial performance. Their primary focus, however, is on external parties. In other words, management accounting helps you keep your business running while financial accounting tells the outside world how well you're running it.

Management accounting

Management accounting, also known as managerial accounting, plays a key role in helping managers carry out their responsibilities. Because the information that it provides is intended for use by people who perform a wide variety of jobs, the format for reporting information is flexible. Reports are tailored to the needs of individual managers, and the purpose of such reports is to supply relevant, accurate, timely information that will aid managers in making decisions. In preparing, analyzing, and communicating such information, accountants work with individuals from all the functional areas of the organization—human resources, operations, marketing, etc.

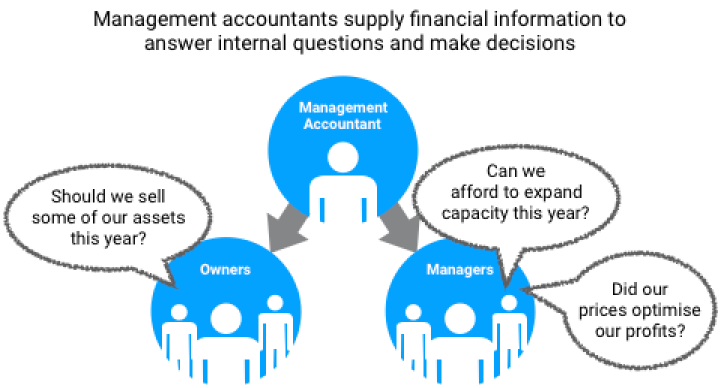


Figure 21: The role of Managerial accounting

Financial accounting

Financial accounting is responsible for preparing the organization's financial statements—including the income statement, the statement of owner's equity, the balance sheet, and the statement of cash flows—that summarize a company's past performance and evaluate its current financial condition. If a company is traded publicly on a stock market such as the NASDAQ, these financial statements must be made public, which is not true of the internal reports produced by management accountants. In preparing financial statements, financial accountants adhere to laws and/or international standards.

While companies headquartered in the United States follow U.S.-based GAAP, many companies located outside the United States follow a different set of accounting principles called International Financial Reporting Standards (IFRS). These multinational standards, which are issued by the International Accounting Standards Board (IASB), differ from U.S. GAAP in a number of important ways.

Who uses financial accounting information?

The users of managerial accounting information are pretty easy to identify—basically, they're a firm's managers. We need to look a little more closely, however, at the users of financial accounting information, and we also need to know a little more about what they do with the information that accountants provide them.

Owners and managers

In summarizing the outcomes of a company's financial activities over a specified period of time, financial statements are, in effect, report cards for owners and managers. They show, for example, whether the company did or didn't make a profit and furnish other information about the firm's financial condition. They also provide some information that managers and owners can use in order to take corrective action, though reports produced by management accountants offer a much greater level of depth.

Investors and creditors

Investors and creditors furnish the money that a company needs to operate, and not surprisingly, they want to know how that business is performing. Because they know that it's impossible to make smart investment and loan decisions without accurate reports on an organization's financial health, they study financial statements to assess a company's performance and to make decisions about continued investment.

Figure 22: Warren Buffet, Presidential Medal of Freedom recipient in 2011

According to the world's most successful investor, Warren Buffett, the best way to prepare yourself to be an investor is to learn all the accounting you can. Buffett, chairman and CEO of Berkshire Hathaway, a company that invests in other companies, turned an original investment of $10,000 into a net worth of $66 billion[30] in four decades, and he did it, in large part, by paying close attention to financial accounting reports.

[30] Forbes Magazine (2016). "The Richest Person in Every State: Warren Buffett." Forbes.com. Retrieved from: http://www.forbes.com/profile/warren-buffett/

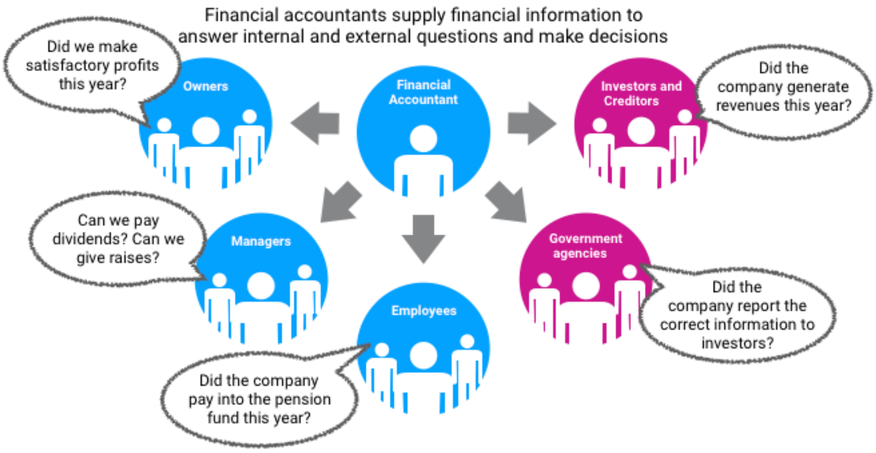


Figure 23: The role of Financial accounting

Government agencies

Businesses are required to furnish financial information to a number of government agencies. Publicly-owned companies, for example—the ones whose shares are traded on a stock exchange—must provide annual financial reports. Companies must also provide financial information to tax authorities.

Other users

A number of other external users have an interest in a company's financial statements. Suppliers, for example, need to know if the company to which they sell their goods is having trouble paying its bills or may even be at risk of going under. Employees and labor unions are interested because salaries and other forms of compensation are dependent on an employer's performance.

Figure 21 and Figure 23 illustrate the main users of management and financial accounting and the types of information produced by accountants in the two areas. In the rest of this chapter, we'll learn how to prepare a set of financial statements and how to interpret them. We'll also discuss issues of ethics in the accounting communities and career opportunities in the accounting profession.

Understanding financial statements

We hope that, so far, at least one thing is clear: If you're in business, you need to understand financial statements. The law no longer allows high-ranking executives to plead ignorance or fall back on delegation of authority when it comes to responsibility for a firm's financial reporting. In a business environment tainted by episodes of fraudulent financial reporting and other corporate misdeeds, top managers are now being held responsible for the financial statements issued by the people who report to them. Top managers need to know how well the company is performing. Financial information helps managers identify signs of impending trouble before it is too late.

The function of financial statements

Put yourself in the place of Connie in Figure 17.5 on the next page, who runs Connie's Confections out of her home. She loves what she does, and she feels that she's doing pretty well. In fact, she has an opportunity to take over a nearby store at very reasonable rent, and she can expand by getting a modest bank loan and investing some more of her own money. So it's decision time for Connie: She knows that the survival rate for start-ups isn't very good, and before taking the next step, she'd like to get a better idea of whether she's actually doing well enough to justify the risk. The basic financial statements will give her some answers.

Since this reader is for an introductory course, we will focus our attention on the income statement and balance sheet only, even though we mentioned other financial statements earlier in the chapter.

Toying with a business idea

To bring this concept closer to home, let's assume that you need to earn money while you're in college and that you've decided to start a small business. Your business will involve selling stuff to other college students, and to keep things simple, we'll assume that you're going to operate on a "cash" basis: you'll pay for everything with cash, and everyone who buys something from you will pay in cash.

You may have at least a little cash on you right now—some currency, or paper money, and coins. In accounting, however, the term **cash** refers to more than just paper money and coins. It also refers to the money that you have in checking and savings accounts and includes items that you can deposit in these accounts, such as money orders and different types of checks.

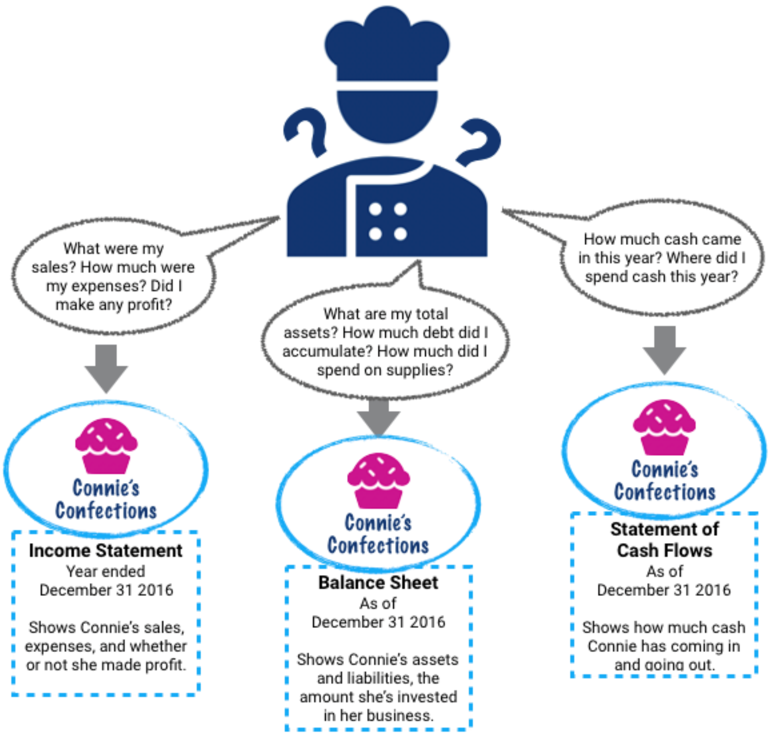


Figure 24: Connie has questions about her business that financial statements can help her answer

Your first task is to decide exactly what you're going to sell. You've noticed that with homework, exams, social commitments, and the hectic lifestyle of the average college student, you and most of the people you know always seem to be under a lot of stress. Sometimes you wish you could just lie back between meals and bounce a ball off the wall. And that's when the idea hits you: Maybe you could make some money by selling a product called the "Stress-Buster Play Pack." Here's what you have in mind: you'll buy small toys and other fun stuff—instant stress relievers—at a local dollar store and pack them in a rainbow-colored plastic treasure chest labeled "Stress-Buster."

The accounting equation

To begin keeping track of your company financially, you'll first need to understand the fundamental accounting equation:

Assets = Liabilities + Owner's equity

Think of assets as things *owned* by your business – cash in the bank, product inventory, etc. And think of liabilities as the amounts *owed* – perhaps you've had a job where your pay check came a couple of weeks after you did the work; during that unpaid window, the amount due to you was a liability to your employer. *Owner's equity* represents the value of the firm according to your financial statements; obviously it is good to own more than you owe.

This simple but important equation highlights the fact that a company's **assets** came from somewhere: either from investments made by the owners (**owner's equity**) or from loans (**liabilities**). This means that the asset section of the balance sheet on the one hand and the liability and owner's-equity section on the other must be equal, or **balance**.

Let's say you have EUR 200 in cash and borrow EUR 400 from your parents and plan to buy a month's worth of plastic treasure chests and toys. After that, you'll use the cash generated from sales of Stress-Buster Play Packs to replenish your supply. You open a bank account for your new business and create your opening financial statement – the **balance sheet.**

The balance sheet

A **balance sheet** reports the following information:

- **Assets**: the resources from which it expects to gain some future benefit
- **Liabilities**: the debts that it owes to outside individuals or organizations
- **Owner's equity**: the investment in the business

At the time you open the account, your balance sheet would look like this:

Stress-Buster Company Balance Sheet As of September 1, 2019	
Assets	
Cash	**600**
Liabilities and Owner's Equity	
Liabilities	400
Owner's Equity	200
Total Liabilities and Owner's Equity	**600**

Figure 25: Stress-Buster's balance sheet as of September 1, 2019

The amount you owe your parents is a liability to you, and your own investment of EUR 200 in the business is represented by your owner's equity.

Now it is time to start buying toys, repackaging them, and selling your Stress-Busters. Each plastic chest will cost EUR 1.00, and you'll fill each one with a variety of five simple toys, all of which you can buy for EUR 1.00 each.

You plan to sell each Stress-Buster Play Pack for EUR 10 from a rented table stationed outside a major dining hall. Renting the table will cost you EUR 20 a month. In order to make sure you can complete your school work, you decide to hire fellow students to staff the table at peak traffic periods. They'll be on duty from noon until 2:00 p.m. each weekday except Fridays, and you'll pay them a generous EUR 7.50 an hour. Wages, therefore, will cost you EUR 240 a month (2 hours × 4 days × 4 weeks = 32 hours × EUR 7.50). Finally, you'll run ads in the college newspaper at a monthly cost of EUR 40. Thus your total monthly costs will amount to EUR 300 (EUR 20 + EUR 240 + EUR 40).

The income statement

Let's say that during your first month, you sell one hundred play packs. Not bad, you say to yourself, but did I make a profit? To find out, you prepare an income statement showing **revenues**, or sales, and **expenses**—the costs of doing business. You divide your expenses into two categories:

- **Cost of goods sold**: the total cost of the goods that you've sold
- **Operating expenses**: the costs of operating your business except for the costs of things that you've sold.

Now you need to do some subtracting:

- The difference between sales revenue and cost of goods sold is your **gross profit**, also known as **gross margin**.
- The difference between gross profit and operating expenses is your **net income** or **profit**, which is the proverbial "bottom line." Note we've assumed you're making money, but businesses can also have a net loss.

Figure 26is your income statement for the first month. (Remember that we've made things simpler by handling everything in cash.)

Stress-Buster Company Income Statement Month Ended September 30, 2019		
Sales (100xEUR 10.00)		EUR 1,000
Less cost of goods sold (100xEUR 6)		600
Gross profit (100x (EUR 10 -EUR 6))		400
Less operating expenses		
Salaries	240	
Advertising	40	
Table rental	20	

	300
Net income (Profit) (EUR 400-EUR 300)	EUR 100

Figure 26: Stress-Buster's income statement for September 2019

Did you make any money?

What does your income statement tell you? It has provided you with four pieces of valuable information:

You sold 100 units at EUR 10 each, bringing in **revenues** or **sales** of EUR 1,000.

Each unit that you sold cost you EUR 6—EUR 1 for the treasure chest plus 5 toys costing EUR 1 each. So your **cost of goods sold** is EUR 600 (100 units × EUR 6 per unit).

Your **gross profit**—the amount left after subtracting cost of goods sold from sales—is EUR 400 (100 units × EUR 4 each).

After subtracting **operating expenses** of EUR 300—the costs of doing business other than the cost of products sold—you generated a positive **net income** or **profit** of EUR 100.

Whereas your **balance sheet** tells you what you have *at a specific point in time*, your **income statement** tells you how much income you earned *over some period of time*, in this case, the month of September.

Companies prepare financial statements on at least a twelve-month basis—that is, for a **fiscal year** which ends on December 31 or some other logical date, such as June 30 or September 30. Fiscal years can vary because companies generally pick a fiscal-year end date that coincides with the end of a peak selling period; thus a crabmeat processor might end its fiscal year in October, when the crab supply has dwindled. Most companies also produce financial statements on a quarterly or monthly basis. For Stress-Buster, you'll want to prepare them monthly to stay on top of how your new business is doing. Let's prepare a new balance sheet to how things have changed by the end of the month.

Recall that Stress- Buster earned $100 during the month of September and that you decided to leave these earnings in the business. This $100 profit increases two items on your balance sheet: the assets of the company (its cash) and your investment in it (its owner's equity). Figure 27 shows what your balance sheet will look like on September 30. You now have $700 in cash: $400 that you borrowed plus $300 that you've invested in the business (your original $200 investment plus the $100 profit from the first month of operations, which you've kept in the business).

Stress-Buster Company Balance Sheet As of September 30, 2019	
Assets	
Cash (original $600 plus $100 earned)	$700
Liabilities and Owner's Equity	
Liabilities	400

Stress-Buster Company Balance Sheet As of September 30, 2019	
Owner's Equity ($200 invested by owner plus $100 profits retained)	300
Total Liabilities and Owner's Equity	$700

Figure 27: Stress-Buster's balance sheet at the end of September 2019

Breakeven analysis

Let's take a short detour to see how Stress Buster's financial information might be put to use. As you look at your first financial statements, you might ask yourself: is there some way to figure out the level of sales you need to avoid losing money—to "break even"? This can be done using **breakeven analysis**. To break even (have no profit or loss), your total sales revenue must exactly equal all your expenses (both variable and fixed). **Variable costs** depend on the quantity produced and sold; for example, each Stress-Buster includes the treasure chest and the toys inside. **Fixed costs** don't change as the quantity sold changes; for example, you'll pay for your advertising whether you sell Stress-Busters or not. The balance between revenue and expenses will occur when gross profit equals all other (fixed) costs. To determine the level of sales at which this will occur, you need to do the following (using data from the previous example):

1. Determine your total fixed costs:
    - Fixed costs = EUR 240 salaries + EUR 40 advertising + EUR 20 table = EUR 300
2. Identify your variable costs on a per-unit basis:
    - Variable cost per unit = EUR 6 (EUR 1 for the treasure chest and EUR 5 for the toys)
3. Determine your **contribution margin** per unit: selling price per unit – variable cost per unit:
    - Contribution margin = EUR 10 selling price – EUR 6 variable cost per unit = EUR 4
4. Calculate your breakeven point in units: fixed costs / contribution margin per unit:
    - Breakeven in units = EUR 300 fixed costs / EUR 4 contribution margin per unit = 75 units

Your calculation means that if you sell 75 units, you'll end up with zero profit (or loss) and will exactly break even. To test your calculation, you can prepare a what-if income statement for

75 units in sales (your breakeven number). The resulting statement is shown in Figure 28

Of course you want to do better than just break even, so you could modify this analysis to a targeted level of profit by adding that amount to your fixed costs and repeating the calculation. Breakeven analysis is rather handy. It enables you to determine the level of sales that you must reach to avoid losing money and the level of sales that you have to reach to earn a certain profit. Such information will be vital to planning your business.

Stress-Buster Company Income Statement Month Ended September 30, 2019 (at breakeven level of sales=75 units)		
Sales (75xEUR 10.00)		EUR 750
Less cost of goods sold (75xEUR 6)		450
Gross profit (EUR 75x (EUR 10 -EUR 6))		300
Less operating expenses		
Salaries	240	
Advertising	40	
Table rental	20	
	300	
Net income (Profit) (EUR 300-EUR 300)		EUR 0

Figure 28: Stress-Buster's breakeven income statement

Financial statement analysis

Now that you know a bit about financial statements, we'll spend a little time talking about they're used to help owners, managers, investors, and creditors assess a firm's performance and financial strength. You can glean a wealth of information from financial statements, but first you need to learn a few basic principles for "unlocking" it.

Types of financing used by companies

Before we go any further, let's outline two basic forms of financing – i.e., how do companies get the money they need in order to operate? One way is to borrow the money, which is known as *debt financing*. A business might take a loan from a commercial bank, or it might issue bonds which pay a particular rate of interest over a set period of time. At the end of the life of the bond, the borrower would repay the *principal*, i.e., the amount borrowed, to the holders of those bonds. Another form of financing would be to sell an ownership stake in the company, which is known as *equity financing*. Many business owners are reluctant to part with an ownership stake in the company because they then have to share the profits with those who have purchased a share of the company. However, lenders will only provide so much financing before they begin to get concerned about the borrower's ability to repay, so in practice, most businesses use some combination of debt and equity financing to fund the operations of the company.

Trend analysis from the income statement

Now let's look at some of the things we can learn from analyzing financial statements. Figure 29is an abbreviated financial statement for Apple for 2014 taken directly from their website. You will note that instead of showing only the current year's results, the company has shown data for the prior two years as well.

From this relatively simple exhibit, considerable information about Apple's performance can be obtained. For example:

- Apple sales grew at 6.95% from 2013 to 2014, not bad for a company with such a large base of sales already, but certainly not the rapid-growth company it once was.
- Net income as a percent of sales (a ratio also known as return on sales) was 21.6% – or in other words, for every $5 in sales, Apple turned more than $1 of it into profit. That is substantial!

Many other calculations are possible from Apple's data, and we will look at a few more as we explore ratio analysis.

Apple Inc. – Consolidated statement of operations (Income statement)
(In millions, except number of shares which are reflected in thousands and per share amounts)

Years ended	September 27, 2014	September 28, 2013	September 29, 2012
Net sales	$182,795	$170,910	$156,508
Cost of sales	$112,258	$106,606	$87,846
Gross margin	$70,537	$64,304	$68,662
Operating expenses:			
Research and development	$6,041	$4,475	$3,381
Selling, general and administrative	$11,993	$10,830	$10,040
Total operating expenses	$18,034	$15,305	$13,421
Operating income	$52,503	$48,999	$55,241
Other income/(expense), net	$980	$1,156	$522
Income before provision for income:			
Taxes	$53,483	$50,155	$55,763
Provision for income taxes	$13,973	$13,118	$14,030
Net income	$39,510	$37,037	$41,733
Earnings per share:			
Basic	$6.49	$5.72	$6.38
Diluted	$6.45	$5.68	$6.31
Shares used in computing earnings per share:			
Basic	$6,085,572	$6,477,320	$6,543,726
Diluted	$6,122,663	$6,521,634	$6,617,483
Cash dividends declared per common share:	$1.82	$1.64	$0.38

Figure 29: Apple statement of operations, 2014

Ratio analysis

How do you compare Apple's financial results with those of other companies in your industry or with the other companies whose stock is available to investors? And what about your balance sheet? Are there relationships on this statement that also warrant investigation? These issues can be explored by using **ratio analysis**, a technique for evaluating a company's financial performance.

Remember that a ratio is just one number divided by another, with the result expressing the relationship between the two numbers. It's hard to learn much from just one ratio, or even a number of ratios covering the same period. Rather, the deeper value in ratio analysis lies in looking at the trend of ratios over time and in comparing the ratios for several time periods with those of other companies. There are a number of different ways to categorize financial ratios.

Here's one set of categories:

- Profitability ratios tell you how much profit is made relative to the amount invested (return on investment) or the amount sold (return on sales).
- Liquidity ratios tell you how well positioned a company is to pay its bills in the near term. Liquidity refers to how quickly an asset can be turned into cash. For example, share of stock is substantially more liquid than a building or a machine.
- Debt ratios look at how much borrowing a company has done in order to finance the operations of the business. The more borrowing, the more risk a company has taken on, and so the less likely it would be for new lenders to approve loan applications.
- Efficiency ratios tell you how well your assets are being managed.

We could employ many different ratios, but we'll focus on a few key examples.

Profitability ratios

Earlier we looked at the return on sales for Apple. Another profitability ratio on which the financial markets focus is earnings per share, also known as EPS. This ratio divides net income by the number of shares of stock outstanding. According to the earlier exhibit, Apple increased its EPS from $5.72 in 2013 to $6.49 in 2014, which indicates growth of about 13% — excellent for a company that is already among the world's largest. Well-paid analysts will spend hours to understand how these results were achieved every time Apple issues new financial statements.

Liquidity ratios

Liquidity ratios are one element of measuring the financial strength of a company. They assess its ability to pay its current bills. A key liquidity ratio is called the current ratio. It simply examines the relationship between a company's current assets and its current liabilities. On September 27, 2014 (remember that balance sheets reflect a point in time), Apple had $68.5 billion in current assets and $63.4 billion in current liabilities. Simply, what this means is that Apple has more money on hand than they need to pay their bills. When a company has a current ratio greater than 1, they are in good shape to pay their bills; companies selling to Apple on credit would not need to worry that it is likely to run out of money.

Apple, Inc. – Consolidated balance sheets
(In millions, except number of shares which are reflected in thousands and par value)

	September 27, 2014	September 28, 2013
Assets:		
Current Assets:		
Cash and cash equivalents	$13,844	$14,259
Short-term marketable securities	$11,233	$26,287
Accounts receivable, net of allowances	$17,460	$13,102
Inventories	$2,111	$1,764
Other current assets	$23,883	$17,874
Total current assets	**$68,531**	**$73,286**
Long-term marketable securities	$130,162	$106,215
Property, plant and equipment, net	$20,624	$16,597
Goodwill and acquired intangible assets, net	$8,758	$5,756
Other assets	$3,764	$5,146
Total assets	**$231,839**	**$207,000**
Liabilities and Shareholders' Equity:		
Current Liabilities:		
Accounts payable	$30,196	$22,367
Accrued expenses	$18,453	$13,856
Other current liabilities	$14,799	$7,435
Total current liabilities	**$63,448**	**$43,658**
Long-term debt	$28,987	$16,960
Other non-current liabilities	$27,857	$22,833
Total liabilities	**$120,292**	**$83,451**
Shareholders' equity:		
Common stock and additional paid-in capital	$23,313	$19,764
Retained earnings	$87,152	$104,256
Accumulated other comprehensive income/(loss)	$1,082	-$471
Total shareholders' equity	**$111,547**	**$123,549**
Total liabilities and shareholders' equity	**$231,839**	**$207,000**

Figure 30: Apple balance sheet, 2014

Apple's current ratio: $\frac{\$68.5\ Billion}{\$63.4\ Billion} = 1.08 > 1$

Now, let's look quickly at something that is not part of the ratio; look down one line on the balance sheet to long-term marketable securities and see that Apple owns $130.2 billion. While they are long term and so not part of the current ratio, these securities are still easily convertible to cash. So Apple has far more cushion than the current ratio reflects, even though it reflected a healthy financial position already.

Debt ratios

A key debt ratio, which tells us how the company is financed, is the **debt-to-equity ratio**, which calculates the relationship between funds acquired from creditors (**debt**) and funds invested by owners (**equity**). For this ratio calculation, we use Apple's *total liabilities*, not just the line on the balance sheet that says long-term debt, because in effect, Apple is borrowing from those who it owes but has not yet paid. Apple's total liabilities at the end of its 2014 fiscal year were $120.3

billion versus owner's equity of $111.5 billion, a ratio of 1.08, which means Apple has borrowed more than it has invested in the business.

Apple's debt to equity ratio: $\frac{\$120.3\ Billion}{\$111.5\ Billion} = 1.08$

To some investors, that high level of debt might seem alarming. But remember that Apple has $130.2 billion invested in marketable securities. If it wished to do so, Apple could sell some of those securities and pay down its debts, thus improving its ratio. It's likely that anyone thinking about lending money to Apple and seeing these figures would be confident that Apple has the ability to pay back what they borrow.

Key takeaways

1. **Accounting** is the process of measuring and summarizing business activities, interpreting financial information, and communicating the results to management and other decision makers
2. **Managerial accounting** deals with information produced for internal users, while **financial accounting** deals with external reporting.
3. The **income statement** captures sales and expenses over a period of time and shows how much a firm made or lost in that period.
4. The **balance sheet** reflects the financial position of a firm at a given point in time, including its assets, liabilities, and owner's equity. It is based on the following equation: assets – liabilities = owner's equity.
5. **Breakeven analysis** is a technique used to determine the level of sales needed to break even—to operate at a sales level at which you have neither profit nor loss.
6. **Ratio analysis** is used to assess a company's performance and financial condition over time and to compare one company to similar companies or to an overall industry.
7. Categories of ratios include: **profitability ratios**, **liquidity ratios**, **debt ratios**, and **efficiency and effectiveness ratios**.

Image credits

Chapter 1

Figure 1: "Steve Jobs." (2011) CC by 2.0. Image retrieved from: https://www.flickr.com/photos/8010717@N02/6216457030

Chapter 2

Figure 10: The Austrian unemployment rate, 1970-2018. Data source: Statistics Austria. Retrieved from StatCube.at

Figure 11: The U.S. Inflation Rate, 1960-2014. Data source: Statistics Austria. Retrieved from: StatCube.at.

Figure 12: CPI Values, 1970-2018. Data source: Statistik Austria. Retrieved from: StatCube.at

Chapter 3

Figure 15: "Apple laptop and notes." Public domain. Retrieved from: https://www.pexels.com/photo/notes-macbook-study-conference-7102/

Figure 16: Dave Mcmt (2009). "A Wendy's in Miles City Montana." CC-BY-2.0. Retrieved from: https://commons.wikimedia.org/wiki/File:Miles_City_MT_-_Wendy%27s.jpg

Figure 17: The U.S. Coast Guard (2010). "The Deepwater Horizon Offshore Drilling Unit on Fire." Public domain. Retrieved from: https://commons.wikimedia.org/wiki/File:Deepwater_Horizon_offshore_drilling_unit_on_fire.jpg

Figure 18: Luis Dantas (2007). "A Samsung desktop SOHO MFP." Public domain. Retrieved from: https://en.wikipedia.org/wiki/Multi-function_printer#/media/File:Multifunctional_Samsung.jpg

Chapter 4

Figure 20: Joe Ravi (2011). "Apple's headquarters at Infinite Loop in Cupertino, California, USA." CC BY-SA 3.0. Retrieved from: https://en.wikipedia.org/wiki/Apple_Inc.#/media/File:Apple_Headquarters_in_Cupertino.jpg

Figure 22: Medill DC (2011). "Medal of Freedom Ceremony." CC BY-NA 2.0. Retrieved from: https://www.flickr.com/photos/medilldc/5448739443/in/photostream/

Figure 29 and Figure 30: Apple Inc. (2015). "Financial Information: 10-K Annual Report 2014." Retrieved from: http://investor.apple.com/financials.cfm

IV Degree-programme-specific section

Banking and Finance

Welcome to the admission procedure
of the bachelor programme Banking and Finance!

We would like to thank you for your interest in our degree programme and would also like to provide you with the following materials as a basis for preparing for our admission procedure.

For the degree-programme-specific test part, you are allowed to use the **calculator** which is **integrated in the exam tool**, as well as a pen and some paper for taking notes. You can familiarise yourself in advance with the functionality of this calculator by taking the trial test. As soon as you have received the access data for the admission test, you will also have access to the trial test. Please make use of this possibility and note that external aids (own calculators, etc.) are **not permitted.**

We hope you enjoy studying the literature, wish you all the best for the admission test and look forward to meeting you in person.

Gernot Kreiger
Degree programme director

Martina Davis
Study programme coordinator

1. Prerequisites

1. Knowledge of the general calculation rules for real numbers (addition, subtraction, multiplication, division)
2. Calculating with fractions, square roots, logarithms
3. Linear equation in two dimensions

Percentage calculation

1% equals 0.01. If p% are added to a size K, K increases to

$K * (1 + p/100)$

Example:

With a value of € 200, a 15% surcharge yields a final price of

$200 * (1 + 15/100) =$ € 230

Discount:

If p% discount is granted, the value of size K decreases to

$K * (1 - p/100)$

Example:

With a value of € 300, a 10% discount yields a final price of

$300 * (1 - 10/100) =$ € 270

Gross/Net

The gross price is made up of the net price of a product by:

Gross price = net price * (1 + Value added tax)

A pair of shoes costs € 100 gross. What is the net price if the sales tax is 16%? What is the amount of VAT?

Net price = 100 / (1 + 0.16) € 86.21

Amount of VAT = 100 – 86.21 = € 13.79.- or 86.21 * 0.16 ≈ € 13.79

2. Equations and inequations

The solutions to the following equation are sought:

4x+7=19

In a first step, 7 is subtracted from both sides.

4x = 12

Now both sides are divided by 4.

x = 3

Basically, all mathematical operations are allowed in an equation. The only condition is to apply this operation to both sides.

Examples:

$$\frac{2x-3}{5-x} = 3 \qquad | * (5-x)$$
$$2x - 3 = 15 - 3x$$
$$5x = 18$$
$$x = \frac{18}{5}$$

$$\frac{1}{x+5} = \frac{2}{x-2} \qquad | * (x+5)(x-2)$$
$$x - 2 = 2x + 10$$
$$x = -12$$

In principle, inequations behave like equations with the caveat that if both sides are multiplied by a negative expression, a ">" becomes a "<" and vice versa, a "<" becomes a ">".

Example:

$$2x > 3 \quad mit \;\; x \in R$$
$$x > \frac{3}{2}, \qquad L = \left\{x \in R | x > \frac{3}{2}\right\}$$

Read: The solution set is the set of all x from the realm of real numbers, for which x is greater than 3/2.

Example:

$8x + 2 > 4x + 1 \qquad (x \in R)$
$8x - 4x > 1 - 2$
$4x > -1$
$x > -\frac{1}{4}, L = \left\{x \in R | x > -\frac{1}{4}\right\} = \left]-\frac{1}{4}, +\infty\right[$

open interval between -1/4 and +infinity.

Example:

$x^2 - 9 > 0 \quad (x \in R)$
$\Leftrightarrow (x-3)(x+3) > 0$
⇒2 cases:
1)
$(x+3) > 0 \;\; and \;\; (x-3) > 0$
$\Rightarrow x > -3 \;\; and \;\; x > 3$
$\Rightarrow L_1 = \{x \in R | x > 3\} =]3, +\infty[$
2)
$(x+3) < 0 \;\; and \;\; (x-3) < 0$
$\Rightarrow x < -3 \;\; and \;\; x < 3$
$\Rightarrow L_2 = \{x \in R | x < -3\} =]-\infty, -3[$

Together: $L_1 \cup L_2 = \{x \in R | (x > 3) \vee (x < -3)\}$

= The union of the two sets of solutions L_1 und L_2
$\vee$…….. or

Similar example (less-than sign instead of greater-than sign!):

$x^2 - 9 < 0 \quad (x \in R)$
$\Leftrightarrow (x-3)(x+3) < 0$

⇒2 cases:

1)
$(x+3) < 0 \;\; and \;\; (x-3) > 0$
$\Rightarrow x < -3 \;\; and \;\; x > 3$
$\Rightarrow L_1 = \{\}$, i.e. there is no solution in R
2)
$(x+3) > 0 \;\; and \;\; (x-3) < 0$
$\Rightarrow x > -3 \;\; and \;\; x < 3$
$\Rightarrow L_2 = \{x \in R | -3 < x < 3\} =]-3, +3[$

together: $L_1 \cup L_2 = \{x \in R | -3 < x < 3\}$

Example:

Two copiers K1 and K2 are available.
K1: costs per copy of € 0.15 monthly; in addition, € 50 maintenance costs

K2: costs per copy of € 0.07, monthly; in addition, € 74 maintenance costs

By what number of monthly copies is K1 cheaper than K2?

$0{,}15x + 50 < 0{,}07x + 74$
$0{,}08x < 24$
$x < 300$

If there are fewer than 300 copies, copy machine K1 is cheaper than K2.

Example:

$\frac{x+1}{x-2} > \frac{1}{3}$

$1. Case:$

$x - 2 > 0, \quad i.e.\ x > 2$

$\frac{x+1}{x-2} > \frac{1}{3} \quad | * (x-2)$

$x + 1 > \frac{1}{3}(x-2) \quad | * 3$

$3x + 3 > x - 2$

$2x > -5$

$x > -\frac{5}{2}$

Both inequations x>-5/2 and x>2 are met for x>2→$L_1 = \{x \mid x > 2\}$

$2. Case:$

$x - 2 < 0, \quad i.e.\ x < 2$

$\frac{x+1}{x-2} > \frac{1}{3} \quad | * (x-2)[negative!]$

$x + 1 < \frac{1}{3}(x-2) \quad | * 3$

$3x + 3 < x - 2$

$2x < -5$

$x < -\frac{5}{2}$

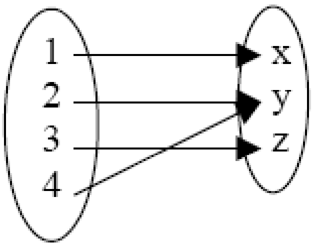
Both inequations x<-5/2 and x<2 are met for x<-5/2→$L_2 = \left\{x | x < -\frac{5}{2}\right\}$

The total solution set is:

$$L = L_1 \cup L_2 = \left\{x | (x > 2) \vee \left(x < -\frac{5}{2}\right)\right\}$$

3. Map or function

By “function” we mean that an x-value can be uniquely assigned to a y-value. Conversely, it does not always work to uniquely assign an x-value to each y-value.



Definition range and range of values (= image range):

The definition range of a function consists of the set of all possible values that can be inserted into the function. Classically, these are all possible x-values that can be used in the function rule.

The range of values denotes all possible values that the function can take (= y-values)

For a straight line:

f: → R, x → k * x + d   or   f(x) = k * x + d   or   y = k * x + d,

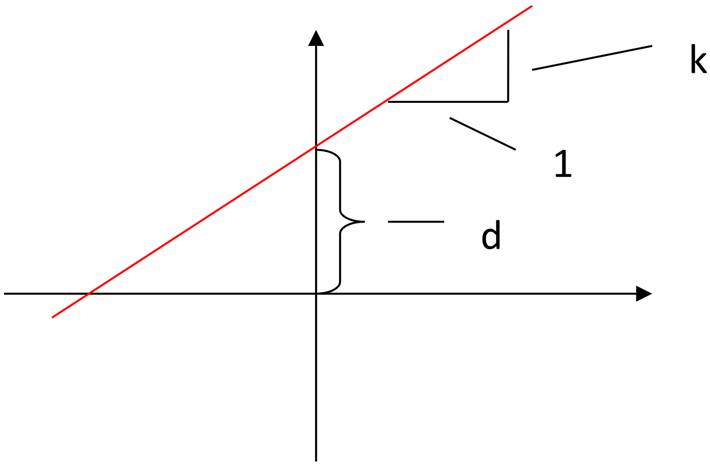
here:

f: the function f

R → R: Mapping of the definition space in R to an image area that is also in R

x → k * x + d : Each value on the x-axis that is in the definition range is assigned a value k * x + d (a y-value).

The constant k represents the slope and d represents the section on the y-axis at x = 0. This representation is reversibly and definitely unique.



In the case of a parabolic equation, however, this does <u>not</u> apply:

f: R → R, x → x^2 or f(x) = x^2 or y = x^2

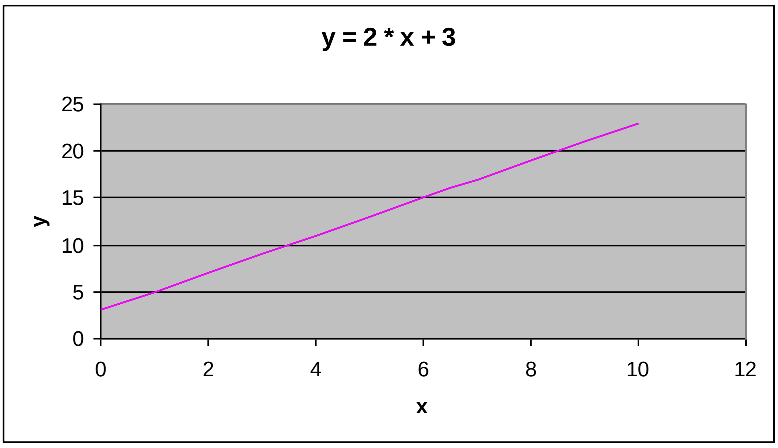
Two x-values can be assigned to each positive y-value:

$$\sqrt{y} = \pm x$$

Functions are represented by means of a value table and a graph:

For a straight line:

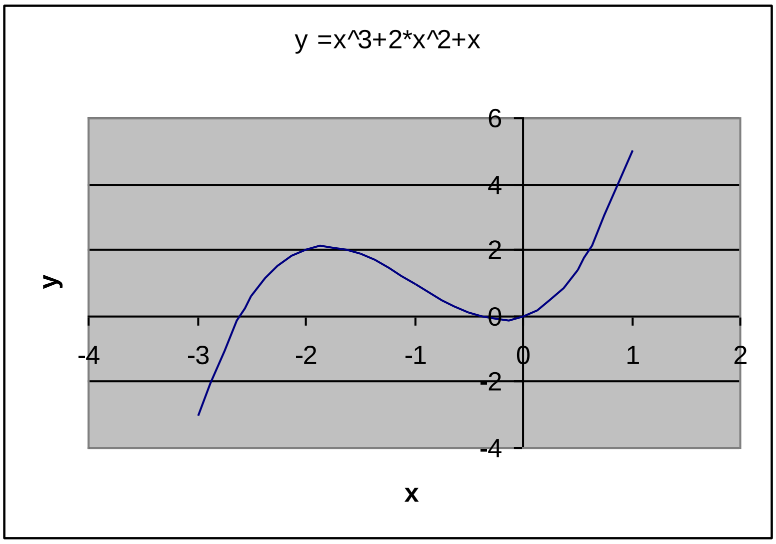
x	y = 2*x+3
0	3
1	5
2	7
3	9
4	11
5	13
6	15
7	17
8	19
9	21
10	23



For a polynomial function:

(x^3 means x^3):

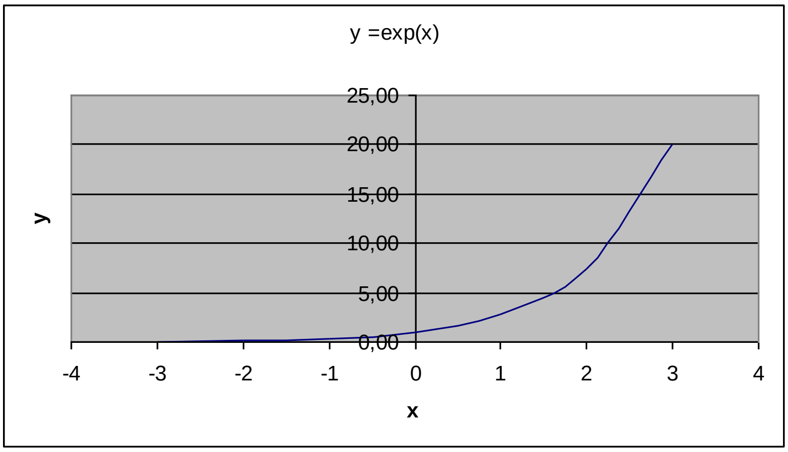
x	y
	=x^3+3*x^2+x
-3	-3
-2.5	0.625
-2	2
-1.5	1.875
-1	1
-0.5	0.125
0	0
0.5	1.375
1	5



For an e-function:

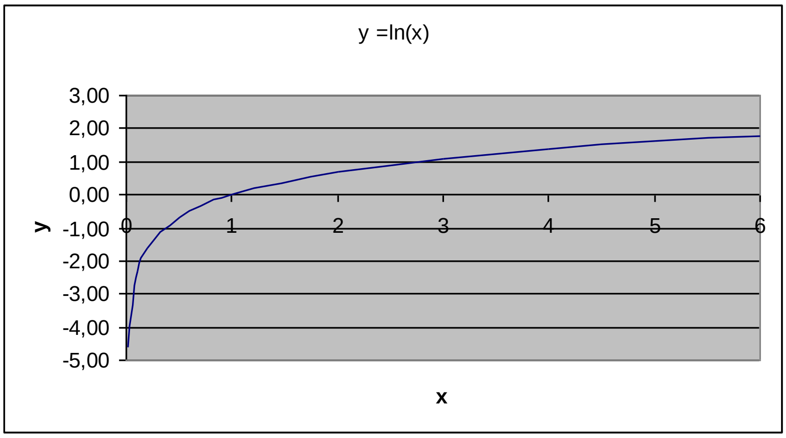
(exp(x) means e^x):

x	y
	=exp(x)
-3	0.05
-2	0.14
-1	0.37
0	1.00
1	2.72
2	7.39
3	20.09



For the natural logarithm:

x	y
	=ln(x)
0.01	-4.61
0.1	-2.30
0.2	-1.61
0.5	-0.69
1	0.00
2	0.69
3	1.10
4	1.39
5	1.61
6	1.79



Example:

1 kg apples costs € 1.80 in a shop. If you buy the same variety from the producer/farmer, the price is € 1 per kg, but expenses of € 30 are incurred for the trip to the producer. What quantity of apples do you have to buy at least in order to make the journey to the producer worthwhile? Represent the facts in a graph.

x kg apples

$$1.8x = 1x + 30$$
$$0.8x = 30 \quad | * 5/4$$
$$x = 37.5kg$$

kg apples	Preis Händler	Preis Erzeuger
x	=1.8*x	=1*x+30
1	1.8	31
2	3.6	32
5	9	35
10	18	40
15	27	45
20	36	50
25	45	55
30	54	60
35	63	65
40	72	70
45	81	75
50	90	80

4. Calculation of interest

4.1. Basics

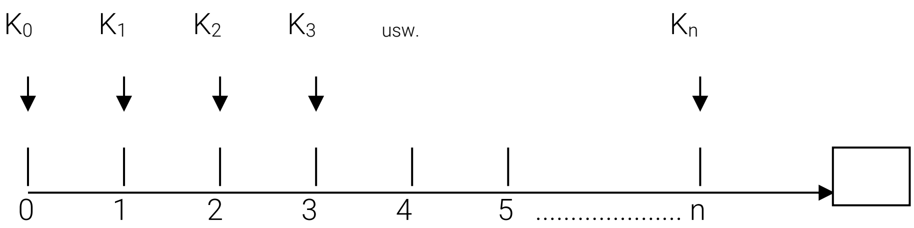
The calculation of interest is similar to a percentage calculation. The main difference is that the interest calculation includes **time.**

Variables that are included in an interest calculation:

1. Interest
2. Capital
3. Interest rate (interest rate)
4. Time (runtime)

There is a variable X which, at time t (t = 0, 1, ..., n) assumes the value Kt (t = 0, 1, 2, ..., n). The values Kt represent a time series that is different from

(t - 1) to t or, in other words, from t to t+1 (1→2, 2→3, 3→4, etc.).



t = 0, 1, 2, ..., n dates

1, 2, ..., n periods (years) n = Term (interest period): years, months, days

Time			
0	=	Beginning of the 1st year	
1	=	Beginning of the 2nd year	= End of the 1st year
....			
N	=	Beginning of (n + 1)th year	= End of n_{th} year

4.1.1. Overview of the methods of interest calculation

In principle, there are 4 methods of interest,

1. Continuous
2. Discrete
3. Simple and
4. Linear.

Type of interest rate	Discount factor	Interest-bearing factor	Interest part
Continuous	$\exp(-i_S * t)$	$\exp(+i_S * t)$	$\exp(i_S * t) - 1$
Discrete	$(1 + i)^{-t}$	$(1 + i)^{t}$	$(1 + i)^{t} - 1$
Simple	$1 / (1 + i * t)$	$1 + i * t$	$i * t$
Linear	$1 - i * t$	$1 + i * t$	$i * t$

Interest surcharges generally only occur at specific times (=interest surcharge dates = interest rate clearing dates = short: interest dates). In the case of discrete interest, the interest surcharge is paid 1x/year (savings book), semi-annually, quarterly or monthly.
With the continuously compounded interest rate, interest is calculated theoretically infinitely many times, i.e. interest is constantly charged. This results in an e-function.
Simple and linear interest rates are often used interchangeably in practice. The individual methods are discussed below in detail.

4.2. Compound interest (exponential interest rate)

The compound interest rate s a discrete interest rate.

There is an initial capital K_0 and interest rate p. At the end of each year t, t = 1,2,3,............., n, interest is paid for K_{t-1}, i.e. not only for K_0 (initial capital), but also for K_0 and also any accrued interest in the meantime.

From an initial capital K_0 you obtain the account balance K_1 after 1 year with p% interest:

$$\underline{K_1} = K_0 + \text{Interest} = K_0 + K_0 * p/100 = K_0 (1 + p/100)^1 = \underline{K_0 (1 + i)^1}$$

This way, it refers to:

p....... Rate of Return, e.g. p = 4

i....... Interest rate (i = p/100), e.g. i = 4% p.a.

The abbreviation "p.a." means "per annum", which stands for "per year" or "yearly". This is the name for the annual interest rate (=nominal annual interest rate).

Interest at the end of each year:

	Accrued Interest	Accrued Interest	Accrued Interest
Z_1 =	$K_0 * i$		
Z_2 =	$K_1 * i$	$= (K_0 + K_0 * i) * i$	$= K_0 * (1+i) * i$
Z_3 =	$K_2 * i$	$K_0 * (1 + i) = K1$ $K_0 * (1 + i) * (1 + i) = K2$ $= K_0 * (1 + i) * (1 + i) * i$	$= K_0 * (1+i)^2 * i$
....			
Z_n =	$K_{n-1} * i$	$= K_0 * (1+i)^{n-1} * i$	$= K_0 * (1+i)^{n-1} * i$

Capital at the end of each year:

$K_1 = K_0 + K_0 * i$	$= K_0 + z$		$= K_0 * (1 + i)$
$K_2 = K_1 + K_1 * i$	$= K_1 + Z_2$	$= K_0 * (1 + i)$ $= K_0 * (1 + i) * (1 + i)$	$= K_0 * (1 + i)^2$
$K_3 = K_2 + K_2 * i$	$= K_2 + Z_3$	$= K_0 * (1 + i)$ $= K_0 * (1 + i) * (1 + i)$ $= K_0 * (1 + i) * (1 + i) * (1 + i)$	$= K_0 * (1 + i)^3$
$K_n = K_{n-1} + K_{n-1} * i$			$K_n = K_0 * (1 + i)^n$

Since the capital increases from period to period, the calculation by means of

$$K_n = K_0 * (1 + i)^n = K_0 * q^n$$

is also called "compounding". The factor

$q = 1 + i = 1 + p/100$

is referred to as the compounding factor. It is often also presented as

$r = 1 + i$.

The interest for the previous year is added to the capital at the end of the year ("capitalized") and with interest in the following year. (Therefore: "Compounding")

Examples:

1. A capital of € 3000 is interest-bearing at 5%. How much do you receive at the end of a year, including interest?
   $K_1 = K_0 * (1 + p/100)^1 = 3000 \cdot (1+5/100) = 3000 \cdot 1.05 =$ € 3150.-

2. A capital of € 3000 is interest-bearing for 4 years at 5% interest. How much is the final amount?
   $K_4 = K_0 * (1 + p/100)^4 = 3000 \cdot (1+5/100)^4 =$ € 3646.52.-

3. How much capital do I have to invest today to obtain € 20,000 after 10 years at a 3% interest rate?
   $K_0 = K_n / (1+p/100)^{10} = 20\ 000 / 1.03^{10} = 14881.88$ €

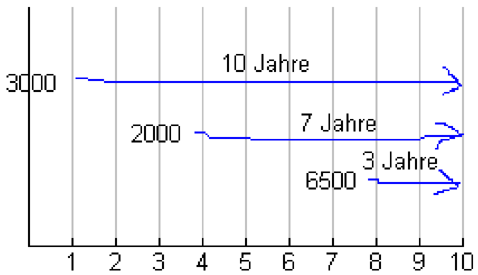
4. At what interest rate was a capital of €4500 invested if it has increased to €5344.59 in 5 years?
   $K_n = K_0 \cdot q^n$
   $5344.59 = 4500 \cdot q^5$ |: 4500
   $1.18769 = q^5$ | 5.Wurzel ziehen
   $q = \sqrt[5]{1,18769} = 1.035$
   $1+p/100 = 1.035 \rightarrow i = 3.5\%$

Given:	$i = 0.08$ Jahr $^{-1}$, $n = 2$ Jahre and $K_0 =$ € 10.000.-
Wanted:	K_2
Solution:	$K_2 = K_0 * (1+ i)^2$
	$= 10.000 * (1 + 0.08)^2$
	= € 11.664.-

5. How long do I have to have a capital of 1000 € in the bank at i = 4.5% p.a. so that I can receive € 2,000?

$K_n = K_0 \cdot q^n$
$K_n / K_0 = q^n$ | ln
$\ln (K_n / K_0) = \ln (q^n)$
$\ln (K_n / K_0) = n * \ln q$
$n = \ln (K_n / K_0) / \ln q$
n = 15.75 years

6. Which final capital do you receive in 10 years if you invest €3,000 today, €2,000 in 3 years, and €6,500 in 7 years? (Interest rate i = 3% p.a.)



A timeline is created for a better overview:
q = 1.03
After that, in principle, only the following is summed up:

$E = 3000 \cdot q^{10} + 2000 \cdot q^7 + 6500 \cdot q^3$ = **€ 13594.22**
If you total the capitals without interest, you receive: 3000+2000+6500 = 11500 €, and so you get **compound interest** of:
13594.22 – 11500 = **€ 2094.22**

(Taken from: Finanzmathematik, Gurtner 2003)

4.3. Discounting

Discounting is the opposite operation. You calculate a value K_0 from a given Kn:

$$K_0 = \frac{K_n}{(1+i)^n}$$

The factor $1/ (1 + i)^n = (1 + i)^{-n}$ is called a discount factor, and the following also applies:

Discount factor = 1 / compounding factor.

With the help of the logarithm, the discrete interest rate can be converted into a continuous one:
$(1 + i)^n = \exp(i_s * n)$
$\ln [(1 + i)^n] = \ln [\exp(i_s * n)]$
$n * \ln (1 + i) = i_s * n$
$\ln (1 + i) = i_s$
where i_s is the continuously compounded interest rate, i.e.:

$K_n = K_0 * (1 + i)^n \rightarrow K_n = K_0 * e^{i_s n}$

The exponential interest rate with annual interest rate i provides the same results for integer interest periods as the equivalent continuously compounded interest rate with i_S.

4.4. Simple (linear) interest rate

With a simple interest rate, the interest per unit of time is always constant. In the simplest case, therefore, the same amount of interest needs to be paid each year. The capital thus grows in a linear fashion with the duration of the investment, which corresponds to a linear equation.

$K_n = K_0 * (1 + n * i) = K_0 + K_0 * i * n$
K_0 = Axis section
$K_0 * i$ = slope of the straight line

For this type of interest rate, the maturity n is not used in the exponent, but as a factor before the interest rate i.

If the initial capital, maturity and interest rate are given, the final capital can be calculated. On the other hand, the initial capital can also be calculated if the final capital, maturity and interest rate are given. Similarly, the maturity and interest rate can be determined if the remaining sizes are known.

$K_n = K_0 * (1 + n*i)$

$$K_0 = \frac{K_n}{(1+n*i)}$$ Simple discounting

$$(1+n*i)$$ Compounding factor

$$v = \frac{1}{(1+n*i)}$$ Simple discount factor

Example:

Linear discount

Given:	$i = 0.10$ year^{-1} , n = 0.5 years und K_n = € 10.000.-		
Wanted:	K_0		
Solution:	$K_0 = K_n * (1- n*i)$		
	= 10.000 * (1 – 0,5 * 0.1)		
	= 10.000 * (0.95) = 9.500 Euro		

In this interest rate method, interest itself is not interest-bearing!

The simple interest rate does not occur in practice for entire years. This calculation is common for parts of an interest period, such as parts of a year.

If i indicates the annual interest rate and if there is one interest date per year, each period < 1 year represents a broken interest period. In this case, interest is usually charged on a linear basis.

There is a certain amount of capital from the beginning of May to the beginning of October:

K_5 is n months (n=5):

$K_5 = K_0 * (1 + 5/12 * i)$

in the future.

For example, if the capital is available from 1 May to 20 May, the following applies:

K_{19} is n days:

$K_{19} = K_0 * (1 + 19/360 * i)$

(Each month is assumed with 30 days and one year with 360 days.)

Another type of discrete interest rate is represented by the sub-annual interest rate

4.5. Sub-annual interest

If, in the case of the interest rate calculation, the accrued interest on the capital is supplemented on several dates (=interest-clearing dates, or interest dates) of the same interval within one year, it is referred to as a sub-annual interest rate, i.e. the interest will not be at the end of each year, i.e. allocated annually to the capital and then with interest, but

semi-annually or
quarterly or
monthly, etc.
or, in general, 1/m -yearly ("one-mtl-year")

The larger m, the smaller the interest period becomes, and the more often interest is added to capital. The annual interest rate corresponds to an exponential interest rate, with the former interest period being not a whole year, but a 1/m – year.

Interest is often paid several times a year. The number of interest periods in the year is m, where m usually takes only the values 2, 4 or 12, which corresponds to interest periods of the duration of a six-month (semester), a quarter or a month.

The **annual nominal interest or nominal annual interest rate** is referred to as j_m.

The index m indicates that the interest rate is calculated m times a year in rates of j_m /m = i_m. It is called **relative interest rate.** For $j_1 = i_1$, short i. Therefore, j_{12} = 6% means that 12 times a year a relative interest rate is charged i_{12} = 0.5%,or j_4 = 4% means that a relative interest rate is charged 4 times a year, i_4 = 1%!

The corresponding symbols for accrued interest are the following: f_m is the annual discount **or the** nominal annual **discount**. f_m / m = d_m is the relative discount **set**. For $f_1 = d_1$, short d

Annual interest	Sub-annual interest
(Given: i =annual nominal interest rate)	(Given: i = annual nominal interest rate)
$K_n = K_0 * (1 + i)^n$	$K_n = K_0 * (1 + i / m)^{n*m}$

Interest	m
Monthly	12
Quarterly	4
Semi-annually	2
Annually	1

The interest rate i_m = i/m is called a relative interest rate: e.g.:
Annual nominal interest rate i = 6% p.a.
Number of interest periods per year m = 12
-> relative interest rate $i_m = i_{12}$= 0.5%.

Example (exponential interest rate):

To which sum does the sum of € 1,500 increase at i = 3% interest p.a. in 10 years if the capital is compounded annually (= the interest period is 1 year)?

$K_n = K_0 * (1 + i)^n = 1,500 * (1 + 0.03)^{10}$ = € 2,015.87

To which sum does the sum of € 1,500 increase at i = 3% interest p.a. in 10 years if the capital is compounded monthly (=the interest period is 1 month, the interest being charged monthly)?

Considerations:
The indication i = 3 % p.a. refers to one year. If interest is paid monthly, this means that in the first month 3/12 % will be added to the initial capital, in the second month 3/12 %, etc.:

$K_{120} = 1{,}500 * (1 + 0.03 / 12)^{120} = 2{,}024.03$

As more interest dates occur, the last result (2,024.03) must be greater than the previous one (2,015.87).
A conforming annual interest rate (=effective interest rate) can be calculated from the relative interest rate: If you set the

$(1 + i / m)^m = 1 + i_{eff}$

so you obtain the effective interest rate i_{eff}.

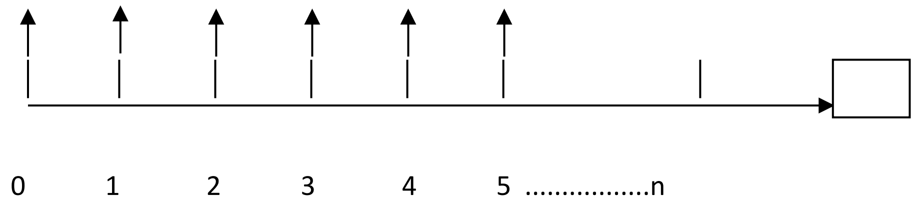
Starting position: 12% p.a. No incidental costs	Annual compounding	Monthly compounding	Quarterly compounding
Nominal interest rate (starting point)	12 % p.a.	12 % p.a.	12 % p.a.
Relative interest rate	(12% / 1 year) = 12%	(12% / 12 months) = 1 %	(12% / 4 months) = 3 %
Effective interest rate	12 %	= actual interest rate for one year when interest is paid monthly $[(1+\frac{i}{m})^m - 1]$ → $[1+ 0.01]^{12} - 1$ = 12.68 % p.a.	= actual interest rate for one year when interest is applied on a quarterly basis $[(1+\frac{i}{m})^m - 1]$ → $[1+ 0.03]^4 - 1$ = 12.55 % p.a.

The effective interest rate is also called the **annual interest rate conforming** to the nominal interest rate (= **equivalent**), i.e. 1% p.m. with monthly interest rate corresponds to a 12.68% annual interest rate (→ 1% p.m. ← 12% p.a.). → The reason for this, in turn, lies in the "interest rate calculation": with monthly interest (based on subsequent interest rates) the interest is added to the initial capital (one speaks of "capitalization") and later the new interest is calculated by the "initial capital incl. interest", etc. The annual nominal interest rate is less than the effective one resulting from a below-year relative interest rate.

In the case of an annual interest rate, capitalisation is made only once, at the end of the year (K_0 = basis for the calculation of interest remains the same throughout the year).

Example:
Annual nominal interest rate i = 12% p.a.
Number of interest periods per year m = 12
Relative interest rate $i_m = i_{12} = 1\%$
$i_{eff} = 12.68\%$

4.6. Anticipative interest yield



This interest rate is referred to as anticipative interest rate if interest is due at the beginning of the interest period. Interest is calculated from the initial capital, just as in the case of the subsequent interest rate. However, just the amount reduced by the interest is paid out. Interest is deducted from the capital, and a discount is granted.
Anticipative interest rates are common when discounting bills of exchange and, in some cases, on borrowings.

Example:
With a principal of € 100,000.00 and an accrued interest rate of d = 10%, € 10,000.00 will be withheld by the credit institution as interest (simple-interest method), and € 90,000.00 will be paid out to the borrower. After a period of e.g. a calendar year, the full principal of € 100,000.00.- must be repaid by the borrower. Up to the term of a whole interest period (here: 1 year), the linear interest rate can be used. Beyond this period, interest is calculated exponentially.

$K_0 = K_n * (1 - n * d) = K_n * (1 - d)^1$

$K_0 = € 100{,}000.00 * (1 - 1 * 0.1)$

$K_0 = € 100{,}000.00 * 0.9$

$\underline{\underline{K_0 = € 90.000.00}}$

€ 90,000 will be paid out at the beginning of the term.

The term is now 5 years. The interest rate remains the same as d = 10%. The question now is how much you get paid out by the credit institution at the beginning of the term. At the end of the term of 5 years, an amount of € 100,000.00.- must still be repaid.

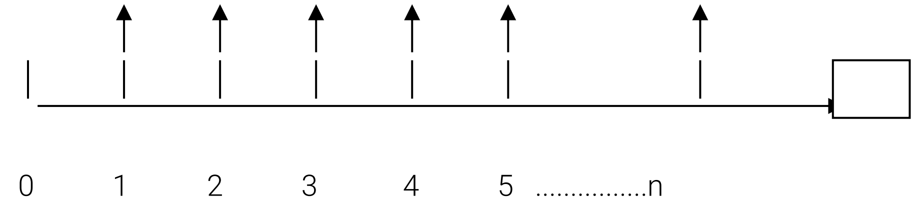
$K_0 = K_n * (1 - d)^n$

$K_0 = € 100{,}000.00 * 0.9^5$

$\underline{\underline{K_0 = € 59{,}049.00}}$

€ 59,049 will be paid out at the beginning of the term.

4.7. Decursive Interest



By decursive interest rate (regular interest rate) we mean the interest is again calculated from the borrowed capital but, in contrast to the anticipative interest rate, is attributed to the capital at the end of the interest period. The formula for this is the following:

$K_n = K_0 * (1 + i * n)$ for linear interest and

$K_n = K_0 * (1 + i)^n$ for the exponential

Example:
The best-known and most common form of simple, decursive interest is the savings book. Let us assume that an amount of € 1,000.00 was available on 15/02/02. The interest rate on 5 years of commitment of the capital is 3.5%. We would like to know now

a. how much this plant is worth at the end of the year and subsequently

b. how much at the end of the five-year term. For the first investment year, we calculate the interest days. We expect 30 days/month and 360 days/year. February has 15 interest-relevant days, and ten months of 30 days are added for the rest of the year:

Solution:

a.

$K_n = K_0 * (1 + i * n)$

$K_n = € 1,000.00 * (1 + 0.035 * 315/360)$

$K_n = € 1,000.00 * 1.030625$

$\underline{\underline{K_n = € 1,030.63}}$

The interest for 315 days is therefore € 30.63.

The savings book will now start next year with a capital of € 1,030.63. This amount of savings, which is increased by the interest, in turn is used to calculate interest, hence the name compound interest.

b.

Since interest rates are exponentially calculated on full periods of interest, after 5 years in the savings book:

$K_n = K_0 * (1 + i)^n$

$K_n = 1,000.00 * (1 + 0.035)^5$

$K_n = 1,000.00 * 1,187686$

$\underline{\underline{K_n = € 1,187.69.-.}}$

4.8. Equivalent interest rates

We have so far used two types of interest, i.e. anticipative and decursive. Which interest rate seems more advantageous?

Interest rates are equivalent if you supply the same final capital from the same initial capital over the same period of time. By equating the anticipative and decursive interest calculation formula, one arrives at:

$d = \frac{i}{1+i}$ Conversion of a decursive interest rate i into an anticipative interest rate d

$i = \frac{d}{1-d}$ Conversion of an anticipative interest rate d into a decursive interest rate i

Example:
In addition to version 1 with an anticipative interest rate of d = 10 %, we also have version 2 with a decursive interest rate of i = 10 % to choose from in a loan offer of € 100,000.00. Which interest rate is more advantageous for you as a borrower now, because it is cheaper? In the case of version 1, we could calculate our payout amount and calculate the interest from the difference with the principal. We do the same in the case of version 2, i.e. the calculation of the final capital with a payout amount of € 100,000.00.- and determine the interest from the difference between the two amounts. We can solve this correct but cumbersome procedure more elegantly with the equivalent interest rates. For version 1, we convert the accrual interest rate d into a decursivei and compare the result with the decursive interest rate of variant 2:

$$i = \frac{0.10}{1 - 0.10}$$

$$i = \frac{0.10}{0.90}$$

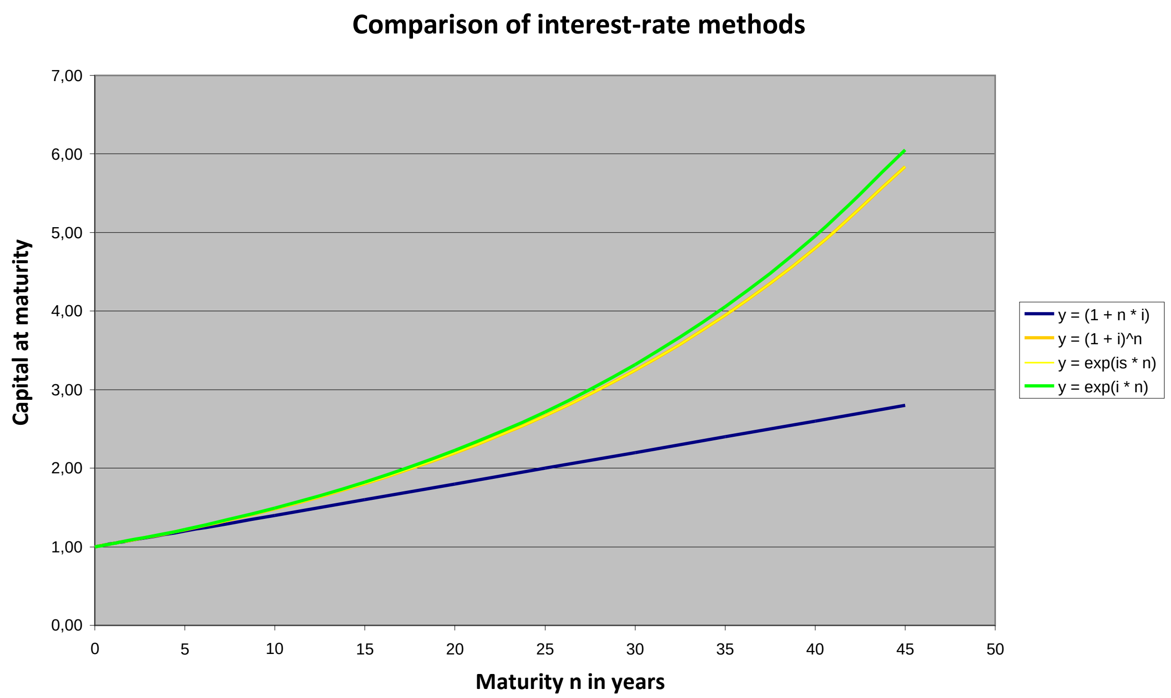
<u>i = 11.11%</u>

That means that an anticipatory interest rate of 10% is a decursive 11.11%, which means that we, as borrowers, have to pay more interest on it. Accordingly, we will opt for version 2 (decursive interest rate) and save 1.11% in decursive interest.

Finally, we make a comparison of the various interest rate methods, using a value table and a line graph.:

Annual nominal interest i =	4%			
Equiv. Continuously compounded i_s =	0.03922071			
	Linear	Discrete	Continuously	Continuously
n	y	y	y	Y
(Years)	= (1 + n * i)	= (1 + i)^n	= exp(i_s * n)	= exp(i * n)
0	1.00	1.00	1.00	1.00
1	1.04	1.04	1.04	1.04
5	1.20	1.22	1.22	1.22

10	1.40	1.48	1.48	1.49
15	1.60	1.80	1.80	1.82
20	1.80	2.19	2.19	2.23
25	2.00	2.67	2.67	2.72
30	2.20	3.24	3.24	3.32
35	2.40	3.95	3.95	4.06
40	2.60	4.80	4.80	4.95
45	2.80	5.84	5.84	6.05



$\ln(1+i) = i_s$

You can see that the discrete and continuously compounded interest rate leave the linear method after some periods of interest far behind.

In general, for one or m interest periods per year, the **equivalence equation** is:

$$1+i=(1+i)^m = \frac{1}{(1-d_m)^m} = \frac{1}{1-d}$$

If a capital is charged at i = 4% or f_4 = 4% or j_2 = 4% for the same time period, we will receive a different final value each time. Annual nominal interest and annual nominal discount, which, on the other hand, provide the same final value, is called equivalent. f_4 = 4% means: the annual nominal discount is 4% p.a. for 4 periods ($d_4 = f_4/4 = 1\%$ per quarter). j_2 = 4% means: the annual interest rate is 4% p.a. at 2 interest periods ($i_2 = j_2/2 = 2\%$ per half-year).
If the nominal interest rates and discount rates i, j_m, f_m, d are equivalent to each other, the relative interest rates and discount rates in i, j_m, f_m, d conform to each other, i.e. they provide the same capital value.

Parameter	expressed by			
	i	r	d	oder
i =	i	r-1	d / (1-d)	d.r
r =	1+i	r	1 / (1-d)	i / d
d =	i / (1+i)	(r+1) / r	d	i / r

Examples:

1) A capital K_0=2,400.- is compounded for n = 5 years at a) i=4%, b) j_4=4%, c) d=4%, d) f_4 =4%.

a) $K_5 = K_0 . r^5 = 2,400 - 1.04^5 = 2,919.97$

b) $K_5 = K_0 . r_4^{20} = 2.400.1.01^{20} = 2,928.46$

c) $K_5 = K_0 . r^5 = 2,400\ 1.04166667^5 = 2,943.44$

d) $K_5 = K_0 . r^{20} = 2,400-1.010\ 101\ 010^{20} = 2,934.32$

2) For property, A offers € 50,000 immediately and € 20,000 in 3 years, B offers € 40,000 immediately. How much more does B have to pay after one year to make his/her offer equivalent to that of the A? j_2 = 4% p.a.

Since the value of a capital depends on the time it is paid, it is not possible to compare the nominal values themselves, but only their values discounted at the same time! We choose the end of the first year as the reference point and start:

$50.000 . 1.02^2 + 20,000 \times 1.02^{-4} = 40,000 - 1.02^2 + X$
$X = 10,000 \times 1.02^2 + 20,000 - 1.02^{-4} = 28,880.91$

If, on the other hand, the end of the third year is chosen as the reference point, we arrive at:

$50.000 . 1.02^6 + 20,000 = 40,000 \times 1.02^6 + X . 1.02^4$

If this equation is divided by 1.02^4 we have the first approach again and realizes that the choice of the reference point can be made at will. We will thus choose the point in time that seems most appropriate for the calculation.

3) The amount K_0=2,400 should be compounded over n =5 years of interest at a) i =4%, b) j_4 =4% taking into account a tax of k=25% (only here in the example):

a) i =4%, $i' = i.(1 - k) = 0.04 . 0.75 = 0.03$

$K_n = 2.400 \cdot 1.03^5 = 2,782.26$

b) j_4 = 4%, i_4 = 1%. Since the capital gains tax is to be paid on the annual interest income, we first calculate the decursive annual interest rate i with the help of the equivalence equation.

$1 + i = (1 + i_4)^4, \quad i = 1.01^4 - 1 = 0.04060401$

$i' = 0.040\ 60401 \cdot 0.75 = 0.0304530075$

$K_5 = 2{,}400 \cdot 1.030453^5 = 2{,}788.38$

4) A machine costs € 16,000, (cash payment). If, on the other hand, if only € 8,000 is paid immediately, in 5 months € 8,500 have to be paid. Which discount rate *d* has been charged?

We apply monthly discounting:

$16{,}000 = 8{,}000 + 8{,}500\ (1 - d_{12})^5$

$d_{12} = 1 - (8{,}500 / 8{,}000)^{1/5} = 0.012\ 051\ 714$; $f_{12} = 14.46\%$

$d = 1 - (1 - d_{12})^{12} = 13.54\%$

4.9. Mixed interest rate

In the case of mixed interest rates, interest on whole years (or interest periods) and for a part of an interest period (typically 1 year) are discounted.

A capital K_0 is n years and T days to be charged.

$K_n = K_0\ (1+ i)^n . (1 + i \cdot T/365)$

in the case of act/365 day count convention. If a capital gains tax of *k*% is taken into account, i is

$i' = i \cdot (1 - k).$

Example:

1. A capital of € 1,000 should be compounded at i = 3.5% from 1 January 1995 to 18 May 1998 (daily counting convention act/365),

(a) without gains tax;

b) with $k = 25\%$.

a) $K_n = 1{,}000 \cdot 1.035^3 \cdot (1 + 0.035 . 138 / 365) = 1{,}123.39$

b) $i' = 0.035 \cdot 0.75 = 0.026\ 25$

$K_n = 1{,}000 \cdot 1.02625^3 . (1 + 0.02625 . 138 / 365) = 1{,}091.56$

2. A capital of € 1,000 was invested on 1 January, 1970. When will it have increased to € 2,500 at

a) i = 4%, b) $j_4 = 4\%$ with mixed interest?

$1{,}000 \cdot 1.04^n = 2{,}500$

$n = \ln 2.5 / \ln 1.04 = 23.362$.

We receive 23 years and from

$2{,}500 = 1{,}000 \cdot 1.04^{23} \cdot (1 + 0.04 \cdot T / 360)$ results in

$T = 128.84 \sim 129$

It will thus have increased to €2,500 on May 9, 1993.

4.10. Day-count convention

For certain types of interest, it is important for the specific calculation to know how many days a month and how many days a year has. The most common day-count conventions are:

30/360, act/360, act/365, act/act.

The simplest case is 30/360, where each month has 30 days and each year 360 (see previous example). The most realistic case is act/act, which determines the current number of days. Depending on the financial product, the daily counting can be different. The admission test will specify the day time counting convention if it differs from 30/360.

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You can download these documents from the website of the European Union (Publications Office) free of charge:https://publications.europa.eu/en/web/general-publications/publications

We hope you enjoy studying the literature, wish you all the best for the admission procedure and look forward to meeting you in person!

Elisabeth Springler
Degree programme director

Michaela Diasek, Andreas Dvorak, Marion Haberfellner
Study programme coordinators

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